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A PROBABILISTIC TREATMENT OF UNCERTAINTY IN NONLINEAR DYNAMICAL SYSTEMS

BY

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A Probabilistic Treatment of Uncertainty in Nonlinear Dynamical Systems

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Abstract

In this work, computationally efficient approximate methods are developed for analyzing uncertain dynamical systems. Uncertainties in both the excitation and the modeling are considered and examples are presented illustrating the accuracy of the proposed approximations.

For nonlinear systems under uncertain excitation, methods are developed to approximate the stationary probability density function and statistical quantities of interest. The methods are based on approximating solutions to the Fokker-Planck equation for the system and differ from traditional methods in which approximate solutions to stochastic differential equations are found. The new methods require little computational effort and examples are presented for which the accuracy of the proposed approximations compare favorably to results obtained by existing methods. The most significant improvements are made in approximating quantities related to the extreme values of the response, such as expected outcrossing rates, which are crucial for evaluating the reliability of the system.

Laplace's method of asymptotic approximation is applied to approximate the probability integrals which arise when analyzing systems with modeling uncertainty. The asymptotic approximation reduces the problem of evaluating a multidimensional integral to solving a minimization problem and the results become asymptotically exact as the uncertainty in the modeling goes to zero. The method is found to provide good approximations for the moments and outcrossing rates for systems with uncertain parameters under stochastic excitation, even when there is a large amount of uncertainty in the parameters. The method is also applied to classical reliability integrals, providing approximations in both the transformed (independently, normally distributed) variables and the original variables. In the transformed variables, the asymptotic approximation yields a very simple formula for approximating the value of SORM integrals. In many cases, it may be computationally expensive to transform the variables, and an approximation is also developed in the original variables. Examples are presented illustrating the accuracy of the approximations and results are compared with existing approximations.

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Chapter 1

Introduction

Many structural and mechanical systems experience vibratory response as a result of environmental loads. Examples include the response of structures to earthquake and wind loadings, vibration of trains and automobiles traveling over rough surfaces, marine structures in waves and aircraft vibration due to turbulence. In all of these cases, there is a great deal of uncertainty in the loads that will be placed on the system over the course of its life.

While detailed knowledge of the forces the structure will be subjected to is not known, some information about the excitation is typically known. For example, measurements from previous earthquakes give engineers an estimate for the magnitude, and sometimes the frequency content, of ground accelerations expected during an earthquake. Similar knowledge is often available for other environmental loads. Although this type of knowledge is useful, the actual excitations are often aperiodic and highly irregular and are not easily characterized. In addition to the complexity of the excitations, the loadings are not repeatable. Time histories recorded from different earthquakes and wind forces measured at different times often look very different.

For these reasons, the excitations are often modeled as stochastic processes and the analysis of systems under such excitation is referred to as random vibration theory. The earliest work in this area was in the 1950's in the aerospace industry. Since then, the methods have been applied to a number of mechanical, civil, and

aerospace systems.

For systems subjected to uncertain excitation, design and performance evaluation measures need to be formulated probabilistically. For example, due to the uncertainties in the excitation, no deterministic bounds can be placed on the magnitude of the response. The main objectives in analyzing systems subjected to uncertain excitations are to determine the probabilistic characteristics of the response and to calculate probabilities related to system performance, such as reliability. The probabilistic characteristics of the response can be determined analytically only for linear systems and a small class of nonlinear systems.

Over the last 40 years, a number of approximate methods have been developed for determining the probabilistic characteristics for the response of nonlinear systems subjected to uncertain excitation. Several of these methods are discussed in chapter 3 and references containing more thorough reviews of the available methods are given. These methods generally provide good estimates to the mean square statistics for nonlinear systems, but often provide poor estimates for quantities related to extreme values of the response, such as the reliability. New approximate methods are presented which are capable of providing good estimates to both the mean square statistics and the reliability for nonlinear systems.

In addition to the uncertainty in the excitation applied to structures, there is also uncertainty in the mathematical modeling of the system. Modeling uncertainty is inherent, as no mathematical model can completely describe the behavior of a physical system. The models developed are typically based on balance laws, experimental observations, or some combination of the two and often contain a number of parameters, such as elastic moduli, damping ratios, natural frequencies, modeshapes, etc. The values of these parameters which will give the best agreement between the response of the model and that of the physical system are not known precisely, and the resulting uncertainty is referred to as *parametric uncertainty.*

One approach to dealing with parametric uncertainty is to take a worst-case approach. In this approach, the parameters are assumed to lie in a bounded domain, the parameter values in this domain giving the worst performance are determined,

and this worst-case performance value is used for design and analysis purposes. One problem with this approach is that it can be highly conservative. This is especially true as the number of uncertain parameters becomes large, since the likelihood of the parameters achieving the worst-case performance may be extremely low. Another difficulty with this approach is finding the bounded domain in which the parameters are assumed to lie. In many cases, it is difficult to put hard bounds on parameter values.

Another approach for dealing with modeling uncertainty is to use probabilistic methods. In order to use probabilistic methods for parametric uncertainty, probability must be interpreted in a Bayesian sense, as a multi-valued logic for plausible reasoning subject to certain axioms (Jeffreys 1961; Box and Tiao 1973; Beck 1989; Beck 1996), since the relative frequency interpretation of probability does not make sense for parametric uncertainty. The probability density function for the parameters represents the relative plausibility of the parameters based on the engineer's knowledge, experience and judgment. One of the problems with using probabilistic methods is the computational difficulties that often arise. Typical problems that arise require computing integrals over the parameter space, which may be high dimensional. Straightforward numerical integration becomes computationally prohibitive when there are more than a few uncertain parameters, and approximate methods are required.

New asymptotic approximations are presented for evaluating the probability integrals arising in the analysis of systems with parametric uncertainty. Approximations are developed for evaluating statistical quantities for the response of systems with parametric uncertainty subjected to uncertain excitation and for evaluating classical reliability integrals. The approximations are all computationally efficient and the accuracy of the methods is demonstrated with a number of examples.

1.1 Organization of Thesis

An overview of the mathematics of stochastic processes and stochastic differential equations is presented in chapter 2, providing the background for the material covered in chapters 3 and 4. The Fokker–Planck equation is introduced and the relationship between stochastic differential equations and the Fokker–Planck equation is illustrated. The concept of reliability and the classical first passage problem are covered, and some issues related to numerical solutions to the Fokker–Planck and backward Kolmogorov equation are discussed.

Chapter 3 contains a review of a number of analytical methods available for investigating the response of structural systems under random excitation. After some background and historical notes on random vibration theory, the chapter begins with a discussion of systems for which analytical solutions can be obtained for the probability distribution of the response. The number of systems for which analytical solutions are available is rather limited and in the remaining sections, a number of approximate techniques based on approximating the stochastic differential equations are reviewed. The review is not intended to be exhaustive, but rather to review some of the well–known methods, with more emphasis being given to those methods which will be used for illustration purposes in chapter 4.

Three new methods for approximating the response of nonlinear dynamical systems to stochastic excitation are presented in chapter 4. The new methods are based on approximating solutions to Fokker–Planck equations and differ from the traditional methods discussed in chapter 3, which are based on approximating solutions to stochastic differential equations. Some illustrative examples are presented and results are compared with those obtained from the methods discussed in chapter 3.

A computationally efficient approach to including modeling uncertainty in the analysis is presented in chapter 5. The model uncertainty is modeled probabilistically, and simple asymptotic formulas are presented for approximating the values of multi–dimensional integrals which arise. Examples are presented which illustrate the accuracy of the asymptotic approximation for computing moments and outcrossing rates for uncertain systems and for evaluating classical reliability integrals.

Chapter 2

Stochastic Processes, Stochastic Differential Equations, Fokker-Planck Equation, and Reliability of Stochastic Dynamical Systems

This chapter contains a brief overview of some of the theory of stochastic processes, with particular emphasis on Markov processes. The Fokker-Planck equation associated with a Markov process is introduced in section 2.3. Some background on stochastic differential equations and stochastic integrals is presented in sections 2.4 – 2.6. In section 2.7, the relationship between stochastic differential equations and the Fokker-Planck equation is illustrated. section 2.8 presents an introduction to reliability for stochastic dynamical systems as well as the classical first passage problem. A review of numerical solutions to the Fokker-Planck equation is given in section 2.9 and some concluding comments are made in section 2.10.

2.1 Stochastic Processes

A stochastic process is an uncertain-valued function for which the uncertainty is described probabilistically. The process will be denoted by $x(t)$ with $t \in T \subset \mathbb{R}$. The parameter t usually represents time and T is the time interval of interest. For each time $t \in T$, $x(t) \in \mathbb{R}^n$ is a random variable. It is assumed that for any finite set of times, $\{t_1, \ldots, tm\}$, with $t_i \in T$, the probability density function for the random

variables $x_i \equiv x(t_i)$

$$p(x_n, t_n; x_{n-1}, t_{n-1}; \dots ; x_1, t_1) \tag{2.1}$$

exists for all $n \in \mathbb{Z}^+$. Knowledge of the probability density functions (2.1) would provide a complete description of the stochastic process.

Note that the probability density functions defined by (2.1) will depend on the mathematical model for the stochastic process. Therefore, the probability density functions should technically be written as

$$p(x_n, t_n; x_{n-1}, t_{n-1}; \dots ; x_1, t_1 | \mathcal{M})$$

where $\mathcal{M}$ denotes the model for the stochastic process. For brevity, this dependence will be dropped in the notation.

Conditional probability density functions can be deduced from (2.1) and Bayes's rule (Feller 1968) by

$$p(x_m, t_m; \dots ; x_k, t_k | x_{k-1}, t_{k-1}; \dots ; x_1, t_1) = \frac{p(x_m, t_m, \dots ; x_1, t_1)}{p(x_{k-1}, t_{k-1}; \dots ; x_1, t_1)}.$$

While this definition is valid regardless of the ordering of the times, the times will be considered to be increasing as the index increases, i.e.,

$$t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_n \leq \cdots .$$

2.2 Markov Processes

A Markov process is a stochastic process for which the future depends only on the present state and not on the previous history of the process or the manner in which the present state was reached. A useful property of Markov processes is that the conditional probability density functions are determined entirely by knowledge of the most recent condition. That is, for all finite sets of times $\{t_1, t_2, \dots , t_{k-1}\}$ and

$\{t_k, t_{k+1}, \ldots, t_n\}$,

$$(2.2)\quad p(x_m, t_m; \ldots; x_k, t_k | x_{k-1}, t_{k-1}; \ldots; x_1, t_1) = p(x_m, t_m; \ldots; x_k, t_k | x_{k-1}, t_{k-1})$$

Equation (2.2) is known as the Markov condition and it implies that all probability density functions can be written in terms of simple conditional probability density functions of the form $p(x, t \,|\, y, s)$ with $s < t$, since given any probability density function $p(x_n, t_n; \ldots; x_1, t_1)$, repeated application of the Markov condition and Bayes' rule gives

$$p(x_n, t_n; \ldots; x_1, t_1) = p(x_1, t_1) \prod_{k=1}^{n-1} p(x_{k+1}, t_{k+1} | x_k, t_k)$$

2.2.1 The Lindeberg Condition and Continuity of Stochastic Processes

While stochastic processes can only be described probabilistically, a question of interest is whether or not samples of the process are continuous. The stochastic processes studied in this work are the response of oscillatory systems to stochastic excitation, for which the sample paths are expected to be continuous.

A conditional probability density function is said to satisfy the *Lindeberg condition* on a domain $\mathcal{D} \subset \mathbb{R}^{n+1}$ if for any $\epsilon > 0$

$$(2.3)\qquad \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\|x-y\|>\epsilon} p(x, t + \Delta t \,|\, y, t)\, dx = 0$$

for all $(y, t) \in \mathcal{D}$. It can be shown that if the conditional probability density function for a Markov process satisfies the Lindeberg condition on $\mathbb{R}^{n+1}$, then the sample paths are continuous with probability one (Gihman and Skorohod 1975).

2.3 The Fokker-Planck Equation

The Fokker-Planck equation is a linear partial differential equation which governs the evolution of the conditional probability density functions of a continuous Markov

process. If, in addition to the Lindeberg condition (2.3), the conditional probability density functions of a Markov process satisfy the following for all $\epsilon > 0$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\|x-y\|<\epsilon} (x-y)\, p(x, t+\Delta t \,|\, y, t)\, dx = a(y,t) + O(\epsilon) \tag{2.4a}$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\|x-y\|<\epsilon} (x-y)(x-y)^T\, p(x, t+\Delta t \,|\, y, t)\, dx = D(y,t) + O(\epsilon) \tag{2.4b}$$

uniformly in y, t and ϵ, then the probability density functions will also satisfy the Fokker-Planck equation

$$\frac{\partial p(x, t \,|\, y, s)}{\partial t} = L(x,t)\, p(x, t \,|\, y, s) \tag{2.5}$$

where $L(x,t)$ is the forward Kolmogorov operator defined by

$$L(x,t)\, \psi(x) = -\sum_{i=1}^{n} \frac{\partial \left(a_i(x,t)\psi(x)\right)}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \left(D_{ij}(x,t)\psi(x)\right)}{\partial x_i\, \partial x_j} \tag{2.6}$$

for all $\psi \in \mathcal{C}^2(\mathbb{R}^n)$. The vector $a(y,t)$ is called the *drift vector* of the process and the matrix $D(y,t)$ is called the *diffusion matrix.*

The Fokker-Planck equation is named after the work of Fokker (1915) and Planck (1917) and is often called the Fokker-Planck-Kolmogorov equation due to the contributions of Kolmogorov (1931). For a derivation of the Fokker-Planck equation, consult Gardiner (1994) or Lin and Cai (1995). Further information regarding the Fokker-Planck equation and applications can be found in the books by Risken (1989) and Soize (1994).

2.4 Stochastic Differential Equations

Stochastic differential equations are differential equations containing terms which are modeled as stochastic processes. They were first investigated by Langevin (1908) in the study of Brownian motion, and have since been applied to problems in a number of fields, including engineering, physics, economics, chemistry, and biology.

Stochastic differential equations are often written in the form

$$\frac{dx(t)}{dt} = \dot{x}(t) = f(x,t) + B(x,t)n(t) \tag{2.7}$$

where $x(t), f(x,t) \in \mathbb{R}^n$, $B(x,t) \in \mathbb{R}^{n\times m}$ and $n(t) \in \mathbb{R}^m$ is the stochastic term, which is often assumed to be rapidly fluctuating. The mathematical idealization of such a term is that for $\tau \neq 0$, $n(t)$ and $n(t+\tau)$ are statistically independent. The mean $n(t)$ is usually taken to be zero, since any nonzero mean can be absorbed into $f(x,t)$. The requirements of zero mean and statistical independence can be written as

$$\begin{aligned} E[n(t)] &= 0 \\ E[n(t)n^T(t+\tau)] &= I\delta(\tau) \end{aligned} \tag{2.8}$$

where E denotes mathematical expectation and I is the $m \times m$ identity matrix. Choosing the identity matrix is merely for convenience since any other amplitude can be accounted for in $B(x,t)$. Excitation satisfying the conditions (2.8) is known as *white noise.*

While excitation having the properties (2.8) satisfies the requirement of statistical independence, it gives $n(t)$ an infinite variance. A more realistic model is that

$$E[n(t)n^T(t+\tau)] = I\,\frac{1}{2\tau_c}\exp(-|\tau|/\tau_c)$$

where τ_c is the correlation time of the excitation. This gives statistical independence for $\tau \gg \tau_c$. Then, white noise could be taken as the limit as $\tau_c \to 0$. In practice, it is not easy to evaluate these limits (Gardiner 1994) and an alternative approach is to rewrite (2.7) as an integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s),s)\,ds + \int_{t_0}^{t} B(x(s),s)\,n(s)\,ds. \tag{2.9}$$

It can be shown (Gardiner 1994) that if $n(s)$ satisfies the properties (2.8), then

$$\int_0^t n(s)\,ds = w(t) \tag{2.10}$$

where $w(t)$ is the multivariate Wiener process having properties

$$E[w(t) - w(s)] = 0 \tag{2.11a}$$

$$E\left[(w(t) - w(s))\,(w(t) - w(s))^T\right] = I|t - s| \tag{2.11b}$$

for all $t, s \in \mathbb{R}$. Equation (2.10) shows that $n(t)$ as defined by (2.8) is like the derivative of the Wiener process, but the latter is not differentiable with probability one (Wiener 1923). Therefore, the proper interpretation of (2.7) is as the integral equation (2.9). Introducing

$$dw(t) = w(t + dt) - w(t) = n(t)dt \tag{2.12}$$

the integral equation (2.9) can be rewritten as

$$x(t) = x(t_0) + \int_{t_0}^t f(x(s), s)\,ds + \int_{t_0}^t B(x(s), s)\,dw(s). \tag{2.13}$$

The second integral in (2.13) is a stochastic integral and is defined in the section 2.5.

2.4.1 Some Comments About White Noise

As mentioned earlier, white noise has an infinite variance and a correlation time of zero, which are unrealistic properties for a model of the excitation. However, the assumption of white noise greatly simplifies the computations and can be thought of as an idealization of a model for the excitations likely to be met in practice. Additionally, excitations for which white noise is a poor model can often be expressed indirectly in terms of linearly filtered white noise. Some well-known engineering applications using filtered white noise include the Kanai-Tajimi spectrum for seismic excitation (Kanai 1957; Tajimi 1960), the Pierson-Moskowitz and JONSWAP

spectra for ocean waves (Hu and Zhao 1993) and Davenport's spectrum for wind excitation (Davenport and Novak 1976).

2.5 Stochastic Integrals of Itô and Stratonovich

In section 2.4, it was shown that the proper interpretation of a stochastic differential equation is as an integral equation, involving a term of the form

$$\int_{t_0}^{t} B(x(s), s)\, dw(s). \tag{2.14}$$

Integrals of the form (2.14) are called stochastic integrals and are defined as a kind of Riemann-Stieltjes integral. The time interval $[t_0, t]$ is partitioned into n subintervals $[t_i, t_j]$ with

$$t_0 < t_1 < \cdots < t_{n-1} < t$$

and the integral is defined as a limit of partial sums. However, due to the rapid fluctuations of the Wiener process, the value of the integral depends on where the integrand is evaluated in each subinterval. Two choices have shown to be useful in practice, and the resulting integrals are known as Itô and Stratonovich integrals, based on the work of Itô (1951) and Stratonovich (1963).

2.5.1 The Itô Integral

The Itô stochastic integral is defined by

$$\int_{t_0}^{t} B(x(s), s)\, dw(s) \equiv \underset{n\to\infty}{\text{ms-lim}} \sum_{i=1}^{n} B(x(t_{i-1}), t_{i-1})\,(w(t_i) - w(t_{i-1})) \tag{2.15}$$

where ms-lim is the mean-square limit (Gardiner 1994). Notice that in each subinterval, $b(x(t), t)$ is evaluated at the previous time t_{i-1}, and, by the properties of the Wiener process, $b(x(t_{i-1}), t_{i-1})$ is statistically independent of the increment $w(t_i) - w(t_{i-1})$. This property makes the Itô integral easy to work with in a number

of applications, and it will be seen that the Fokker-Planck equation corresponding to a stochastic differential equation is easily obtained if the integral in (2.13) is an Itô integral. A drawback to the Itô integral is that some of the resulting properties, such as the change of variables formula, are different from ordinary calculus (Soong and Grigoriou 1993).

2.5.2 The Stratonovich Integral

The Stratonovich integral, denoted here by $S\int$, is defined by

(2.16)
$$S\int_{t_0}^{t} b(x(s),s)\,dw(s) \equiv \underset{n\to\infty}{\text{ms-lim}} \sum_{i=1}^{n} B\left(\frac{x(t_i)+x(t_{i-1})}{2}, t_{i-1}\right)(w(t_i)-w(t_{i-1}))\,.$$

Notice that the difference between the Itô integral and the Stratonovich integral is where the integrand is evaluated in each interval. Also notice that if $b(x(s),s)$ is independent of x, the two integrals will be equivalent. The Stratonovich integral has the advantage that many of its properties, including the change of variables formula, are the same as those of ordinary calculus.

2.6 Itô and Stratonovich Stochastic Differential Equations

It was shown in section 2.4 that the proper interpretation of the stochastic differential equation (2.7) is as the integral equation (2.13). The integral equation is often written in differential form as

(2.17)
$$dx(t) = f(x(t),t)\,dt + B(x(t),t)\,dw(t).$$

If the integral in (2.13) is interpreted as an Itô [Stratonovich] integral, the differential equation (2.17) is called an Itô [Stratonovich] stochastic differential equation.

In the remaining chapters, stochastic differential equations will be written in both forms (2.7) and (2.17), depending on which is more convenient for the appli-

cation.

2.6.1 Relation Between Itô and Stratonovich SDE's

In section 2.7, it is shown that the Fokker-Planck equation corresponding to a stochastic differential equation can be obtained easily if the differential equation is thought of as an Itô equation. However, Stratonovich equations are a more natural choice for an interpretation which assumes the excitation is a physical noise with a finite correlation time, which is allowed to become arbitrarily small after calculating desired quantities (Gardiner 1994). Stratonovich equations are also preferred in some applications, since the rules of ordinary calculus can be applied to Stratonovich equations, while the rules of the Itô calculus are different. For these reasons, it is useful to be able to convert an equation of one type into the other type.

It can be shown (Gardiner 1994) that the Itô SDE

$$dx(t) = f^I(x(t), t)\, dt + B(x(t), t)\, dw(t)$$

is equivalent to the Stratonovich SDE

$$dx(t) = f^S(x(t), t)\, dt + B(x(t), t)\, dw(t)$$

where

$$f_i^S = f_i^I - \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{n} B_{kj}\frac{\partial B_{ij}}{\partial x_k}. \tag{2.18}$$

The terms appearing in the summation in equation (2.18) are known as the Wong-Zakai correction terms (Wong and Zakai 1965) and provide a simple conversion between Itô and Stratonovich equations. The formula also shows that if B is independent of x, the two equations are equivalent, as mentioned in section 2.5.2.

2.7 Connection between Stochastic Differential Equations and the Fokker-Planck Equation

An ordinary differential equation is said to be *memoryless* if the solution for future times can be obtained from the present state independently of the manner in which the present state was reached. The solution, $x(t)$, to a memoryless stochastic differential equation of the form (2.17) is a Markov process. Intuitively this is clear since the future response depends only on the present state and not on past values, therefore the state-transition probability density functions should also be independent of the previous state values. A rigorous proof of this result can be found in Arnold (1974). Since the solution to the differential equation is a Markov process, the state-transition probability density functions for $x(t)$ are governed by a Fokker-Planck equation. It is easiest to determine the corresponding Fokker-Planck equation if the differential equation is interpreted as an Itô equation. Stratonovich equations can be handled by converting to the equivalent Itô equation using the Wong-Zakai correction terms (2.18).

It can be shown (Gardiner 1994; Lin and Cai 1995; Caughey 1971) that the response $x(t)$ to an Itô equation

$$dx(t) = f(x(t),t)\,dt + B(x(t),t)\,dw(t),$$

is a Markov process with drift vector $a(x,t) = f(x(t),t)$ and diffusion matrix $D(x,t) = B(x(t),t)B^T(x(t),t)$. Therefore, from (2.5) and (2.6), the state-transition probability density function $p(x,t\,|\,y,s)$ satisfies the following Fokker-Planck equation

$$\begin{aligned}\frac{\partial p(x,t\,|\,y,s)}{\partial t} = &-\sum_{i=1}^{n}\frac{\partial\left(f_i(x,t)p(x,t\,|\,y,s)\right)}{\partial x_i}\\ &+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\frac{\partial^2\left(B_{ik}(x,t)B_{kj}(x,t)p(x,t\,|\,y,s)\right)}{\partial x_i\partial x_j}.\end{aligned} \tag{2.19}$$

As discussed in chapter 3, the time-dependent Fokker-Planck equation is very

difficult to solve, and often the long-term, or steady-state response of the system is of interest. If $f(x,t)$ and $B(x,t)$ do not depend explicitly on t, the steady-state probability density function $p(x)$ satisfies the stationary, or reduced, Fokker-Planck equation

$$-\sum_{i=1}^{n} \frac{\partial \left(f_i(x)p(x)\right)}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m} \frac{\partial^2 \left(B_{ik}(x)B_{kj}(x)p(x)\right)}{\partial x_i \partial x_j} = 0. \tag{2.20}$$

In terms of the forward Kolmogorov operator defined by (2.6) with $a(x) = f(x)$ and $D(x) = B(x)B^T(x)$, the stationary Fokker-Planck equation can be written in the simple form

$$L(x)p(x) = 0. \tag{2.21}$$

Even for the stationary Fokker-Planck equation (2.21), there are very few known solutions for nonlinear systems, as discussed in section 3.1. Some comments on numerical solutions to the Fokker-Planck equation are given in section 2.9 and a number of new methods for approximating solutions to the stationary Fokker-Planck equation are given in chapter 4.

2.7.1 Differential Equations with Memory

While memoryless differential equations can be used to model many physical systems, many systems of engineering interest are hysteretic, i.e., future response depends not only on the present state, but also on what happened to the system in the past. Since analysis of differential equations with memory is much more difficult, a number of higher-order memoryless differential equations have been proposed to model hysteretic behavior, see e.g., Visintin (1994). These models typically introduce auxiliary variables such that when the solutions are projected onto the variables of interest, the response displays some hysteretic qualities. A well-known example in earthquake engineering is the Bouc-Wen model (Bouc 1967; Wen 1980) for hysteresis. Only memoryless differential equations will be studied in this work.

2.8 Reliability and the First Passage Time

One of the most important properties of a dynamical system subjected to stochastic excitation is its reliability. Due to the uncertainty in the excitation, no deterministic bounds can be set on the response amplitude, but the probability that the states remain in a "safe" or "acceptable" domain is often of interest. Typically, a *safe set*, $\mathcal{S}$, and a *failure set*, $\mathcal{F}$, are defined such that the system performance is acceptable if $x \in \mathcal{S}$ and unacceptable if $x \in \mathcal{F}$. The reliability is then defined by

$$R(x_0, T) = P(\, x(t) \in \mathcal{S} \,|x(0) = x_0\,) \quad \text{for all } t \in [0, T]$$

where R is the reliability, $P(\cdot)$ denotes probability and $[0, T]$ is the time interval of interest. Associated with the reliability is the failure probability, P_f, which is given by

$$P_f(x_0, T) = P(\, x(t) \in \mathcal{F} \,|x(0) = x_0\,) \quad \text{for some } t \in [0, T].$$

Clearly, $P_f(x_0, T) = 1 - R(x_0, T)$, provided that the sets $\mathcal{S}$ and $\mathcal{F}$ are complements, as usually defined.

A classical problem associated with reliability theory is the *first passage problem.* Letting τ be the time at which $x(t)$ first leaves $\mathcal{S}$, the first passage problem is to determine the probability distribution for τ, i.e., to determine $P(\tau \geq t)$ for all times $t > 0$. From the above definitions, it can easily be seen that $P(\tau \geq t)$ given that $x(0) = x_0$ is equal to $R(x_0, t)$.

It can be shown (Gardiner 1994) that $R(x_0, t)$ satisfies the *backward Kolmogorov equation*

$$\frac{\partial R(x_o, t)}{\partial t} = L^*(x_0, t) R(x_0, t) \tag{2.22}$$

where $L^*(x_o, t)$, the adjoint of the forward Kolmogorov operator, is the *backward Kolmogorov operator.* If $a(x, t)$ is the drift vector and $D(x, t)$ is the diffusion matrix for the Markov process $x(t)$, the backward Kolmogorov operator is defined for all

$\psi \in \mathcal{C}^2(\mathbb{R}^n)$ by

$$L^*(y,t)\psi(y) = \sum_{i=1}^{n} a_i(y,t)\frac{\partial \psi(y)}{\partial y_i} + \sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij}(y)\frac{\partial^2 \psi(y)}{\partial y_i \partial y_j}. \tag{2.23}$$

Assuming that $\mathcal{S}$ is a simply connected region with boundary ∂S, the initial condition for the backward Kolmogorov equation (2.22) is

$$R(x_0, 0) = 1 \text{ for } x_o \in \mathcal{S} \tag{2.24}$$

and the boundary condition is

$$R(x_0, t) = 0 \text{ for } x_0 \in \partial\mathcal{S} \text{ for all } t > .$$

The first passage problem for second-order systems subjected to white noise excitation was first posed by Yang and Shinozuka (1970) as an initial-boundary value problem for the backward Kolmogorov equation and by Crandall (1970) for the Fokker-Planck equation. Fischera (1960) proved the well-posedness of these problems.

Unfortunately, analytical solutions of the backward Kolmogorov equation exist only for the simplest scalar systems as illustrated by Darling and Siegert (1953). Therefore, even for linear systems, approximate methods are required for estimating the reliability. Approximate methods based on outcrossing rates are discussed in section 3.2.2.

2.9 Numerical Solution of the Fokker-Planck and Backward Kolmogorov Equations

Due to the limited number of analytical solutions available for the Fokker-Planck and backward Kolmogorov equations, a number of approaches have been made to obtain numerical solutions to these equations.

Some of the first numerical solutions obtained were for the first passage time of

a second-order linear oscillator (Chandiramani 1964; Crandall et al. 1966). In their method, the safe region in the state space was divided into cells, and the probability was diffused from cell to cell in each time based on the analytical solution for the state-transition probability density function for the linear oscillator. Later, Sun and Hsu (1988, 1990) developed a generalized cell mapping method to obtain solutions to the first passage problem for nonlinear second-order oscillators. Here, short-time simulations were used to map the probability from cell to cell in each time step.

Another approach to obtaining approximate solutions is based on Galerkin's method. Atkinson (1973) used this method to investigate stationary solutions of the Fokker-Planck for second-order nonlinear systems. The trial functions were based on the eigenfunctions of the forward Kolmogorov operator for the linear systems. The method was extended to investigate nonstationary response (Wen 1975) and Bouc-Wen type hysteretic systems (Wen 1976). A Galerkin approach using locally defined Gaussian probability density functions was developed by Kunert (1991). One of the difficulties in applying this approach is obtaining good trial functions, as discussed in Wen (1975).

Finite element solutions to the stationary Fokker-Planck equation for second-order nonlinear systems have been obtained by Langley (1985) and Langatangen (1991). One of the main difficulties associated with numerically solving the stationary Fokker-Planck equation is satisfying the global normalization condition for the probability density function. An alternative approach based on solution of the Chapman-Kolmogorov equation has been developed by Naess and Johnsen (1991).

A Petrov-Galerkin finite element method has been used by Spencer, Bergman and colleagues to obtain numerical solutions to both the Fokker-Planck equation and the backward Kolmogorov equation for second-order linear and nonlinear oscillators (Spencer and Bergman 1991; Bergman and Spencer 1992; Spencer and Bergman 1993; Bergman et al. 1996) and for some three-dimensional systems (Wojtkiewicz et al. 1995). These methods have been able to obtain accurate solutions for two and three dimensional problems, but a significant amount of supercomputer time is required in order to obtain the solutions.

All of the numerical methods so far proposed require a significant amount of computational time. In addition, the solutions obtained for the probability density function are not typically in a convenient form for calculating statistical quantities of interest, such as moments and stationary outcrossing rates. In chapter 4, efficient methods for approximating the solutions to the stationary Fokker-Planck equation are developed.

2.10 Final Remarks

It has been shown that given any stochastic ordinary differential equation, the Fokker-Planck equation is a (deterministic) partial differential equation governing the evolution of the state transition probability density function. Both stochastic differential equations and the Fokker-Planck equation have been shown to be useful in practice; the following quotation is from Gardiner (1994)

> There are many techniques associated with the use of Fokker-Planck equations which lead to results more directly than by direct use of the corresponding stochastic differential equation; the reverse is also true. To obtain a full picture of the nature of diffusion processes, one must study both points of view.

Much of the past research in approximating the response of nonlinear oscillators to stochastic excitation has been focused on obtaining approximate solutions to stochastic differential equations. Analysis methods based on approximating the stochastic differential equations are reviewed in chapter 3 and new methods based on approximate solutions to the Fokker-Planck equation are presented in chapter 4.

Chapter 3

Random Vibration Theory

Random vibration theory is the study of the vibrational response of mechanical and structural oscillatory systems under uncertain dynamic excitation. The uncertain excitations, for example wind or earthquake loads, are typically modeled as stochastic processes, leading to stochastic differential equations for the response.

Vibration response due to stochastic excitation was first studied in the mid 1950's in the aerospace industry. Fuselage panels near jet engines were experiencing fatigue cracks due to the acoustic excitation from the jet exhausts. When the excitation from the engines was studied, it was found to be rapidly fluctuating, aperiodic, and lacked repeatability from one experiment to the next (Clarkson and Mead 1972). Some other early problems studied were the effects of atmospheric turbulence on aircraft (Press and Houboult 1955) and the reliability of payloads in rocket-propelled vehicles (Bendat et al. 1962). In all of these cases, the response was sufficiently complex and irregular that a probabilistic or statistical approach was found to be much more useful than traditional deterministic approaches.

In the last few decades, random vibration theory has spread from the aerospace industry into a number of engineering fields. Some examples include the response of ships and offshore structures to wave excitation (Grigoriu and Allbe 1986; Leira 1987; Hamamoto 1995; Hijawi et al. 1997), response of civil structures to wind (Davenport and Novak 1976; Kareem 1992; Islam et al. 1992; Chen 1994; Feng and Zhang 1997) and to seismic excitation (Tajimi 1960; Iwan 1974; Wen 1975;

Vanmarcke 1976; Park 1992; Papadimitriou and Beck 1994; Schueller et al. 1994) and vibration of vehicles traveling over bumpy surfaces (Newland 1986; Schiehlen 1986; Ushkalov 1986; Hunt 1996). Several textbooks giving a good overview of the subject have been written, e.g., (Crandall and Mark 1963; Lin 1967; Nigam 1983; Roberts and Spanos 1990; Newland 1993; Soong and Grigoriou 1993; Lin and Cai 1995; Lutes and Sarkani 1997).

In random vibration studies, the system response is a stochastic process, and the goal of the engineer is to determine as much probabilistic and/or statistical information about the process as is possible. If the state-transition probability density functions $p(x, t|x_0, t_0)$ could be obtained for all times $t > t_0$, all probabilistic and statistical information about the system could be determined from the probability density functions and the system's initial conditions. Unfortunately, for most nonlinear systems of interest, there is no known way to determine the state-transition probability density function, or even the stationary probability density function $p(x)$. A summary of systems for which analytical solutions to the Fokker-Planck equation are known is given in section 3.1. Often, statistical parameters for the response, such as moments and expected outcrossing rates, are of interest, as discussed in section 3.2. section 3.3 presents some approximate methods which have been developed based on approximating the stochastic differential equations.

3.1 Exact Solutions in Random Vibration Theory

For some dynamical systems, it is possible to obtain an analytical solution to the Fokker-Planck equation for the system. The solution to the nonstationary Fokker-Planck equation can be obtained for linear systems of any dimension subjected to additive Gaussian white noise excitation. For nonlinear systems, analytical solutions are known only for some special systems in one state variable (Risken 1989). Solutions to the stationary Fokker-Planck equation are known for a limited class of nonlinear single degree-of-freedom oscillators. Some solutions are available for nonlinear multi-degree-of-freedom oscillators, but the solutions typically require special

relationships between the system and excitation parameters which are unlikely to be met in practice. A more complete review of the known solutions can be found in Lin and Cai (1995).

3.1.1 Linear Systems with Gaussian White Noise Excitation

The state-transition probability density function can be obtained for time-invariant linear systems of any dimension subjected to additive Gaussian white noise (or linearly filtered Gaussian white noise). Such linear dynamical systems under additive stochastic excitation can be written in the form

$$dx(t) = Ax(t)\,dt + B\,dw(t)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $w(t) \in \mathbb{R}^m$ is the standard Wiener process having the properties in (2.11a) and (2.11b). In this case, the state-transition probability density function $p(x,t|x_0,t_0)$ can be obtained by solving a (deterministic) Lyapunov matrix differential equation (Lin 1967).

Without loss of generality, t_0 can be taken to be zero and the transition probability density function is given by

$$p(x,t|x_0) = \frac{1}{(2\pi)^{n/2}\sqrt{\det P(t)}} \exp\left(-\frac{1}{2}(x-x_0)^T P^{-1}(t)(x-x_0)\right)$$

where $P(t)$ is the solution to the differential Lyapunov equation

$$\begin{aligned} \dot{P} &= AP + PA^T + BB^T \\ P(0) &= P_0. \end{aligned}$$

Here, $P_0 = E[x_0 x_0^T]$ is the covariance matrix for the initial state x_0. If the linear ordinary differential equation $\dot{x} = Ax$ is stable, then the solution of the Lyapunov differential equantion approaches a steady-state value $\hat{P}$ as $t \to \infty$. The stationary

covariance matrix $\hat{P}$ can be obtained by solving the algebraic Lyapunov equation

$$A\hat{P} + \hat{P}A^T + BB^T = 0$$

and the stationary probability density function is given by

$$p(x) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det \hat{P}}} \exp\left(-\frac{1}{2}x^T\hat{P}^{-1}x\right).$$

3.1.2 Stationary Solutions for Nonlinear Systems

Exact solutions to the stationary Fokker-Planck equation have been obtained for a number of single degree-of-freedom nonlinear oscillators. The first solutions obtained (Andronov et al. 1933) were for single degree-of-freedom oscillators with linear damping and nonlinear stiffness. Solutions for more general nonlinear single degree-of-freedom oscillators, including systems with energy-dependent damping were obtained by Caughey and Ma (1982). The class of systems with known solutions was extended through the concept of detailed balance by Yong and Lin (1987) and further generalized by Lin and Cai (1988) through a generalization of Stratonovich's method of stationary potential.

The single degree-of-freedom systems with energy-dependent damping that will be of interest in later sections can be written in the form

$$\ddot{x} + f(H)\dot{x} + g(x) = \sqrt{2D}\,n(t) \tag{3.1}$$

where

$$H(x,\dot{x}) = \frac{\dot{x}^2}{2} + \int_0^x g(\xi)\,d\xi \tag{3.2}$$

is the Hamiltonian and $n(t)$ is Gaussian white noise. If the following technical conditions are met: $H(x,\dot{x})$, $f(H) \in \mathcal{C}^2$, $H(x,\dot{x}) \geq 0$, $\exists H_o$ such that $H \geq H_o \Rightarrow f(H) > 0$, and $f'(H)/f^2(H) \to 0$ as $H \to \infty$, then the solution to

the stationary Fokker-Planck equation associated with (3.1) is

$$p(x,\dot{x}) = a \exp\left(-\frac{1}{D}\int_0^{H(x,\dot{x})} f(\eta)\,d\eta\right) \tag{3.3}$$

where a is a normalization constant (Caughey and Ma 1982).

There are very few solutions to the stationary Fokker-Planck equation for multi-degree-of-freedom systems. Even for multi-degree-of-freedom analogs of single degree-of-freedom systems with known solutions, solutions cannot typically be found, and in the few cases where analytical solutions are known, these solutions typically require restrictive relationships between the system and excitation parameters unlikely to be met in practice. Further information on known solutions for multi-degree-of-freedom systems can be found in the books by Soize (1994) and Lin and Cai (1995).

Unfortunately, even for single degree-of-freedom nonlinear oscillators, many of the systems of interest are not of the solvable form. Although the known solutions are not directly applicable to these systems, they have been very helpful in testing the accuracy of proposed approximation methods. Additionally, these solutions can be used to approximate the solution of other nonlinear systems, as in sections 3.3.2, 4.3 and 4.4.

3.2 Statistical Parameters of Interest

3.2.1 Moments

Some of the most important properties of a stochastic process are characterized by its moments, particularly the first and second moments. If $x(t)$ is a scalar stochastic process with probability density function $p(x,t)$, the n^{th}-order moment of $x(t)$, denoted $m_n(t)$ is defined by

$$m_n(t) = \int_{-\infty}^{\infty} x^n p(x,t)\,dx = E[x^n(t)].$$

Similarly, for vector processes, joint moments can defined by

$$m_{ijk\ldots} = E[x_1^i x_2^j x_3^k \ldots].$$

Much of the information about a stochastic process is contained in the first and second order moments. For example, for Gaussian distributed processes, all probabilistic and statistical information can be determined from knowledge of the first and second-order moments. The first moments give the mean values of the response and the second moments give the mean square values, which typically provide a measure of the average energy in the system. Additionally, knowledge of the first two moments of a stochastic process enable upper bounds to be placed on the reliability of the response through the generalized Tchebycheff's inequality. If the random process $x(t)$ has mean $\mu_x(t)$ and variance $\sigma_x^2(t)$ and the derivative of $x(t)$ has variance $\sigma_{\dot{x}}^2(t)$, the generalized Tchebycheff's inequality (Lin 1967) gives

(3.4)

$$P(\,|x(t) - \mu_x(t)| \geq \epsilon \text{ for some } t \in [0,T]\,) \leq \frac{1}{2\epsilon^2}\left(\sigma_x^2(0) + \sigma_x^2(T)\right) + \frac{1}{\epsilon^2}\int_0^T \sigma_x(t)\sigma_{\dot{x}}(t)dt$$

for all $\epsilon > 0$. The left-hand side in (3.4) is the probability of failure associated with the safe set $\mathcal{S}(t) = \{x \in \mathbb{R} : |x - \mu_x(t)| < \epsilon\}$. While (3.4) is useful as an upper bound, it is often highly conservative in practice.

3.2.2 Expected Outcrossing Rates and Reliability Estimation

While mean square values provide a lot of information about the response, often the primary goal is to determine the reliability of the system. As discussed in section 2.8, reliability is the probability that the response variables remain in a safe or acceptable domain during a time interval of interest. In vibration applications, the safe domain is often chosen to be a region where displacements stay within some prescribed limits.

For a given safe region, the expected outcrossing rate is the mean rate at which the response leaves the safe region into the unsafe region. For second-order os-

cillatory systems subjected to stationary random excitation, there are well-known methods for estimating the reliability based on expected outcrossing rates. The original work in this area was done by Rice (1944) and a number of extensions have been developed since then.

Consider a single degree-of-freedom oscillator subjected to stationary white noise excitation

$$\ddot{x} + f(x, \dot{x}) + g(x) = \sqrt{2D}\, n(t).$$

In one of the simplest cases, the safe domain is the region $\mathcal{S} = \{(x,\dot{x}) \in \mathbb{R}^2 : x < b\}$ for a given $b > 0$. Letting $p(x,\dot{x})$ be the stationary probability density function for the Markov process $x(t)$, the expected crossing rate of the threshold b is given by Rice's formula (Rice 1944)

$$\nu = \int_0^\infty \dot{x} p(b, \dot{x}) d\dot{x}. \tag{3.5}$$

Typically, $b \gg \sqrt{E[x^2]}$ so that threshold crossings are rare and the resulting failure probability is low. If the threshold crossings are assumed to arrive independently, it follows that the threshold crossings are Poisson distributed in time and the probability of failure is given by

$$P_f(t) = 1 - \exp(-\nu t) \tag{3.6}$$

from which the reliability is given by $R = \exp(-\nu t)$. Equation (3.6) was first suggested by Coleman (1959) for the reliability of structures against first-excursion failures.

The most questionable aspect of these results is the assumption that the crossings arrive independently, see, for example, Bogdanoff and Kozin (1961). It has been shown by Cramer (1966) that if x is normally distributed, then the threshold crossings are asymptotically Poisson distributed as $b \to \infty$, and there is some evidence to suggest that this is true for non-Gaussian distributed variables as well (Dunne and

Wright 1985; Roberts 1978a; Roberts 1978b). However, for finite values of b, it is well-known that the threshold crossings do not arrive independently, and that they tend to occur in "clusters" (Lin 1967). Despite this shortcoming, equation (3.6) is still useful as efficient way to get an order-of-magnitude estimate of the failure probability.

3.3 Approximate Methods Based on Stochastic Differential Equations

Due to the limited number of analytical solutions available for nonlinear systems under stochastic excitation, a number of approximate methods have been developed. In this section, some of the well-known methods based on approximation of the stochastic differential equations are presented, with more attention given to those methods which are used later for illustrating the new approximate methods developed in chapter 4.

3.3.1 The Method of Equivalent Linearization

The most popular method used in the analysis of nonlinear systems is the method of equivalent linearization. It was originally developed by Booton (1954) and Caughey (1959a, 1959b) for single degree-of-freedom systems and was later generalized for multi degree-of-freedom systems (Foster 1968; Iwan and Yang 1972; Iwan 1973; Atalik and Utku 1976).

Single Degree-of-Freedom Systems

In the method of equivalent linearization, the response of the nonlinear stochastic differential equation of interest

$$\ddot{x}_{nl} + f(x_{nl}, \dot{x}_{nl}) = \sqrt{2D}\, n(t) \tag{3.7}$$

is approximated by that of a linear system

$$\ddot{x}_{lin} + \beta_{eq}\dot{x}_{lin} + \omega_{eq}^2 x_{lin} = \sqrt{2D}\, n(t). \tag{3.8}$$

The parameters of the linear system (3.8) are selected to provide the best approximation to the nonlinear system (3.7). To do this, the mean square equation error is minimized, i.e. β_{eq} and ω_{eq}^2 solve

$$\min_{\beta_{eq},\omega_{eq}^2} E\left[\left(f(x_{lin},\dot{x}_{lin}) - (\beta_{eq}\dot{x}_{lin} + \omega_{eq}^2 x_{lin})\right)^2\right]. \tag{3.9}$$

Performing the minimization, the optimal parameters are found to be

$$\beta_{eq} = \frac{E\left[\dot{x}_{lin} f(x_{lin},\dot{x}_{lin})\right]}{E\left[\dot{x}_{lin}^2\right]} \tag{3.10a}$$

$$\omega_{eq}^2 = \frac{E\left[x_{lin} f(x_{lin},\dot{x}_{lin})\right]}{E\left[x_{lin}^2\right]}. \tag{3.10b}$$

If $n(t)$ is modeled as Gaussian white noise with properties given by (2.8), the stationary probability density function for the linear system can easily be obtained as

$$p(x_{lin},\dot{x}_{lin}) = \frac{\beta_{eq}\,\omega_{eq}}{2\pi D} \exp\left(-\frac{\beta_{eq}\,\omega_{eq}^2}{2D}\, x_{lin}^2 - \frac{\beta_{eq}}{2D}\, \dot{x}_{lin}^2\right).$$

The probability density function for the nonlinear system is then approximated by that of the linear system.

Multi Degree-of-Freedom Systems

The multi degree-of-freedom analog of (3.7) is

$$M\ddot{x}_{nl} + f(x_{nl},\dot{x}_{nl}) = B\, n(t) \tag{3.11}$$

where $x(t) \in \mathbb{R}^n$, $n(t) \in \mathbb{R}^m$, $B \in \mathbb{R}^{n\times m}$, M is a symmetric, positive-definite $n \times n$ matrix, and $f(x,\dot{x}) \in \mathbb{R}^n$. As in the single degree-of-freedom case, the nonlinear

equation is replaced by a linear system

$$M\ddot{x}_{lin} + C_{eq}\dot{x}_{lin} + K_{eq}x_{lin} = B\,n(t) \tag{3.12}$$

so that the mean square equation error is minimized. To minimize the error, the $n \times n$ matrices C_{eq} and K_{eq} are chosen to solve

$$\min_{C_{eq},K_{eq}} E\left[\|f(x_{lin},\dot{x}_{lin}) - C_{eq}\dot{x}_{lin} - K_{eq}x_{lin}\|^2\right] \tag{3.13}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$. Using the identity that for $x, y \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$,

$$\frac{\partial}{\partial A}\langle Ax, y\rangle = x\,y^T$$

and differentiating (3.13) with respect to C_{eq} and K_{eq} gives

$$\begin{pmatrix} K_{eq} & C_{eq} \end{pmatrix} = P^{-1}\begin{pmatrix} E[f(x_{lin},\dot{x}_{lin})\,x^T] & E[f(x_{lin},\dot{x}_{lin})\,\dot{x}^T] \end{pmatrix} \tag{3.14}$$

where

$$P = \begin{pmatrix} E[x_{lin}\,x_{lin}^T] & E[x_{lin}\,\dot{x}_{lin}^T] \\ E[\dot{x}_{lin}\,x_{lin}^T] & E[\dot{x}_{lin}\,\dot{x}_{lin}^T] \end{pmatrix}.$$

Since P is a function of K_{eq} and C_{eq}, equation (3.14) contains $2n^2$ coupled, nonlinear equations for the unknown elements of K_{eq} and C_{eq}. A simple iterative procedure is available in the case of Gaussian white noise excitation.

Iterative Procedure for Multi Degree-of-Freedom Systems

When $n(t)$ is modeled as Gaussian white noise, P can be obtained as the solution of the algebraic Lyapunov equation

$$A_{eq}P + PA_{eq}^T + BB^T = 0 \tag{3.15}$$

where

$$A_{eq} = \begin{pmatrix} 0 & I \\ -M^{-1}K_{eq} & -M^{-1}C_{eq} \end{pmatrix}.$$

The iterative procedure is as follows

1. Start with an initial estimate of P
2. Use this P in (3.14) to obtain K_{eq} and C_{eq} (and hence A_{eq})
3. Use A_{eq} from step 2 and solve (3.15) for P
4. Repeat steps 2 and 3 until convergence

The probability density function for the nonlinear system is again approximated by that of the linear system. Letting $y_{nl} = (x_{nl}^T, \dot{x}_{nl}^T)^T$, the stationary probability density function is given by the multidimensional Gaussian probability density function with covariance matrix P

$$p(y_{nl}) \approx \frac{1}{(2\pi)^n \sqrt{\det P}} \exp\left(-\frac{1}{2} y_{nl}^T P^{-1} y_{nl}\right).$$

Note that in the iteration procedure, step 2 involves evaluating expectations and matrix multiplication and step 3 requires solution of a linear equation. Both of these steps can be done efficiently, and, except for simulation methods (discussed in section 3.3.4), this is basically the only method that has been able to obtain approximations for nonlinear systems in many dimensions.

3.3.2 Approximation by Nonlinear Systems

The equivalent linearization method can be easily and efficiently applied to many nonlinear systems of interest. The method generally gives reasonably good approximations to mean square values, even for systems with large nonlinearities. However, the approximate probability density function obtained is Gaussian, while the response of nonlinear systems is known to be non-Gaussian. This can lead to large

errors when approximating quantities related to extreme values of the process, such as reliability or stationary outcrossing rates.

In an effort to obtain better accuracy than that obtained by the method of equivalent linearization, some equivalent nonlinearization methods have been developed. The basic idea for equivalent nonlinearization was originally suggested by Caughey and particular problems have been investigated by Lutes (1970), Kirk (1974), and Caughey (1986). A special case of equivalent nonlinearization in which computations can be done rather efficiently, termed partial linearization (Elishakoff and Cai 1993), has since been developed. In these methods, the nonlinear differential equation is approximated by a different nonlinear system which has a known stationary probability density function. Since the approximate system is nonlinear, the approximate probability density function obtained will be non-Gaussian, and the hope is that this will lead to better approximations, particularly for reliability. The applicability of these methods is primarily limited to single degree-of-freedom systems, since the approximate system must be one for which the stationary Fokker-Planck equation can be solved.

Partial Linearization

In the method of partial linearization (Elishakoff and Cai 1993), the response of the nonlinear single degree-of-freedom system

$$\ddot{x}_{nl} + f(x_{nl}, \dot{x}_{nl}) + g(x_{nl}) = \sqrt{2D}\, n(t) \tag{3.16}$$

is approximated by the response of the nonlinear system with linear damping

$$\ddot{x}_{plin} + \beta_{eq}\dot{x}_{plin} + g(x_{plin}) = \sqrt{2D}\, n(t). \tag{3.17}$$

If the excitation, $n(t)$, is modeled as Gaussian white noise, the stationary probability density function for the response of (3.17) is

$$p(x_{plin}, \dot{x}_{plin}) = a \exp\left(-\frac{\beta_{eq}}{D} G(x_{plin}) - \frac{\beta_{eq}}{2D}\dot{x}_{plin}^2\right) \tag{3.18}$$

where a is a normalization constant and

$$G(x_{plin}) = \int_0^{x_{plin}} g(\xi)\, d\xi. \tag{3.19}$$

As in the case of equivalent linearization, the equivalent damping parameter, β_{eq} is given by minimizing the mean square equation error

$$\min_{\beta_{eq}} E\left[\left(f(x_{plin}, \dot{x}_{plin}) - \beta_{eq}\dot{x}\right)^2\right].$$

Performing the minimization gives

$$\beta_{eq} = \frac{E[\dot{x}_{plin}\, f(x_{plin}, \dot{x}_{plin})]}{E[\dot{x}_{plin}^2]} \tag{3.20}$$

and the probability density function for the nonlinear system is approximated by (3.18) with β_{eq} as given by (3.20).

For single degree-of-freedom systems, this method can be applied efficiently, since obtaining the optimal parameter only requires computing expectations. Also notice that the formula (3.20) for the optimal damping parameter is the same as the formula obtained by the method of equivalent linearization (3.10a).

Equivalent Nonlinearization

In the method of equivalent nonlinearization, the nonlinear system chosen to approximate (3.16) is of the form

$$\ddot{x}_{eqnl} + f_{eqnl}(H)\dot{x}_{eqnl} + g(x_{eqnl}) = \sqrt{2D}\, n(t) \tag{3.21}$$

where H is the Hamiltonian as given in (3.2) and $f_{eqnl}(H)$ is a specified function of the Hamiltonian. Note that the partial linearization method is a special case of this method, obtained by choosing $f_{eqnl}(H) = \beta_{eq}$. The stationary probability density function for the equivalent nonlinear system (3.21) is

$$p(x_{eqnl}, \dot{x}_{eqnl}) = a \exp\left(-\frac{1}{D}\int_0^H f(\eta)\, d\eta\right).$$

Typically, $f_{eqnl}(H)$ is taken to be a polynomial in H and the coefficients of the polynomial are chosen to minimize the mean square equation error. For example, if $f_{eqnl}(H) = \sum_{i=1}^{p} \theta_i H^{i-1}$, then the minimization condition is

$$\min_{\theta} E\left[\left(f(x_{eqnl}, \dot{x}_{eqnl}) - \dot{x}_{eqnl}\sum_{i=1}^{p}\theta_i H^{i-1}\right)^2\right]. \tag{3.22}$$

This results in a set of nonlinear algebraic equations for the parameters which usually need to be evaluated numerically. Note that the expectations which need to be evaluated are with respect to the probability density function for the equivalent nonlinear system, and numerical integration is often required to evaluate the expectations. Thus, at each iteration in the minimization procedure, numerical integration is required to evaluate the expectations, making this method more computationally expensive than either the method of equivalent linearization or the method of partial linearization.

3.3.3 Closure Techniques

It can be shown (Soong and Grigoriou 1993) that a system of (deterministic) ordinary differential equations can be written for the moments of a Markov process which satisfies a Fokker-Planck equation. These equations cannot generally be solved, since they form an infinite hierarchy and solution of any finite set of these equations involves too many unknowns to be solved. Therefore, a number of methods have been developed to approximate the relationship between higher moments and lower moments in order to obtain a finite set of equations which can be solved. Such methods

are referred to as closure methods.

In one of these methods, Gaussian closure, only second-order moments are determined, and the relationship between higher-order and lower-order moments is assumed to be the same as for Gaussian probability density functions. It can be shown that this method gives the same results as equivalent linearization. Non-Gaussian closure techniques were first introduced by Crandall (1980). One of the most frequently used non-Gaussian closure techniques is the cumulant-neglect closure method. Here, the equations for the cumulants are obtained, and all cumulants above a certain order are assumed to be zero. This yields a system of nonlinear, algebraic equations for the unknown moments, which can be solved numerically. The number of equations to solve grows very rapidly as the dimension of the state is increased as well as when the order of the approximation is increased, making this method computationally expensive for multi-degree-of-freedom systems.

While these methods are often able to provide better estimates to the moments than the method of equivalent linearization, the methods do not provide any estimate of the probability density function for the system, and hence provide no way to approximate outcrossing rates or reliability. Some methods have been proposed to determine an approximate probability density function based on the moments, including Edgeworth series (see, e.g., Roberts and Spanos (1990)) and the principle of maximum entropy (Trebicki and Sobczyk 1996), but little work has been done to determine the accuracy of such methods for determining reliability.

A similar approach is to take a parameterized Gram-Charlier series, consisting of a series of Hermite polynomials multiplying a Gaussian probability density function, and determine the parameters to satisfy a certain number of moment equations (Soong and Grigoriou 1993). This approach is especially unfavorable for computing reliability estimates, as the approximation can result in negative probabilities over regions of the response.

3.3.4 Other Methods

A number of other approximate methods based on approximating the stochastic differential equations have been developed, including perturbation methods, stochastic averaging, and dissipation energy balancing. More details on these methods can be found in the books by Lin and Cai (1995), Soong and Grigoriou (1993), and Roberts and Spanos (1990).

Another very important class of methods used in the analysis of stochastic dynamical systems are simulation methods, including Monte Carlo simulation, importance sampling, and other related methods. In these methods, the response of the system is computed for a large number of samples of the excitation and the desired statistics are computed based on the samples. Although the methods generally require a considerable amount of computation time, they are very useful for approximating the response statistics of multi-degree-of-freedom systems, since, although the computational time for each sample increases with the number of degrees of freedom, the number of samples required is virtually independent of the dimension of the system. However, in order to obtain accurate estimates for statistics related to extreme values, such as outcrossing rates, the required number of samples is often very large.

Chapter 4

Approximate Methods for Random Vibrations Based on the Fokker-Planck Equation

All of the methods presented in chapter 3 were based on writing stochastic differential equations for the response variables of interest and approximating these equations with other equations for which known solutions to the corresponding Fokker-Planck equation exist. In this chapter, approximate methods are developed based on approximating the Fokker-Planck equation directly.

In the first two methods presented, equivalent systems are found whose stationary probability density functions provide the best fit to the Fokker-Planck equation for the nonlinear system of interest. In the third method, the approximate probability density functions are chosen based on the given system and do not correspond to any "equivalent system". Examples are presented to illustrate each of the methods.

4.1 Overview of the Methods

The goal of the methods presented in this chapter is to obtain probabilistic and/or statistical quantities of interest for a system governed by a nonlinear stochastic differential equation of the form

$$dx(t) = f(x)\,dt + B\,dw(t). \tag{4.1}$$

While the methods developed in this chapter are applicable to systems with parametric excitation (where $B = B(x)$), all of the examples considered will have only additive excitation. One reason for this is that for systems under parametric excitation, the main concern is typically stochastic stability or bifurcation and it is generally accepted that linearization techniques are unsuitable for studying these aspects of dynamic response (Roberts and Spanos 1990). Another reason is that the approximate probability density functions chosen are all based on solutions to systems under additive excitation, and are not expected to provide good approximations for systems with parametric excitation. For more details on systems with parametric excitation, see (Ibrahim 1985; Falsone 1992; Yoon and Ibrahim 1995; Katafygiotis et al. 1997; DiPaola and Falsone 1997).

The state-transition probability density function for the response of the system (4.1) satisfies the Fokker-Planck equation

$$\frac{\partial p(x,t|x_0,t_0)}{\partial t} = L_{nl}(x)\, p(x,t|x_0,t_0) \tag{4.2}$$

where $L_{nl}(x)$ is the forward Kolmogorov operator associated with the nonlinear system (4.1). Note that $L_{nl}(x)$ is a *linear* operator; the "nl" subscript is used to denote that $L_{nl}(x)$ is the Kolmogorov operator corresponding to the nonlinear system (4.1). If (4.2) could be solved for the state-transition probability density function $p(x,t|x_0,t_0)$, all probabilistic and statistical information could be obtained from the probability density function. Unfortunately, solving the time dependent Fokker-Planck equation is extremely difficult and there are no known solutions for nonlinear systems in more than one state variable, as discussed in section 3.1. There have been some numerical solutions to (4.2) in two and three dimensions, but the solutions require substantial computational time, as discussed in section 2.9.

This work will be restricted to finding approximations to stationary probability density functions and associated quantities of interest, such as moments and

expected outcrossing rates. The stationary Fokker-Planck equation is given by

$$L_{nl}(x)p(x) = 0. \tag{4.3}$$

In addition to the Fokker-Planck equation (4.3), the stationary probability density function must satisfy the boundary condition

$$p(x) \to 0 \text{ as } \|x\| \to \infty$$

and normalization condition

$$\int_{\mathbb{R}^n} p(x)\, dx = 1.$$

Even for the stationary Fokker-Planck equation, there are very few nonlinear systems for which known solutions exist. In addition, the boundary condition and the global normalization condition are not particularly amenable for numerical solutions.

Since there is no known way to solve the stationary Fokker-Planck equation (4.3) for general nonlinear systems, approximate methods will be developed. The approximate methods presented herein are based on finding a probability density function $p(x)$ for which

$$L_{nl}(x)p(x) \approx 0$$

in some sense.

To do this, a set of parameterized probability density functions

$$\mathcal{P} = \{ f \in \mathcal{C}^2(\mathbb{R}^n) : f(x) = p(x|\theta),\ \theta \in \Theta \subset \mathbb{R}^p \}$$

is chosen, where $\mathcal{C}^2$ is the space of twice continuously differentiable functions and $p(x|\theta)$ is a probability density function parameterized by $\theta \in \Theta \subset \mathbb{R}^p$. For example, in a one-dimensional problem, $\mathcal{P}$ could be taken to be the set of Gaussian probability

density functions with zero mean and variance θ^2

$$\mathcal{P} = \left\{ f \in \mathcal{C}^2(\mathbb{R}) : f(x) = \frac{1}{\sqrt{2\pi}\theta} \exp\left(-\frac{x^2}{2\theta^2}\right), \theta \in \mathbb{R}^+ \right\}.$$

The criterion for making $L_{nl}(x)p(x|\theta) \approx 0$ is chosen as

$$\min_{f \in \mathcal{P}} \|L_{nl}(x)f(x)\|$$

or, equivalently,

$$\min_{\theta \in \Theta} \|L_{nl}(x)p(x|\theta)\| \tag{4.4}$$

where $\|\cdot\|$ is a norm on $\mathcal{C}^2(\mathbb{R}^n)$. The various methods will differ primarily in the set $\mathcal{P}$ of probability density functions used and the choice of the norm.

4.1.1 Selection of the Set $\mathcal{P}$

While there is a lot of freedom in selecting the set of approximate probability density functions, there are some natural choices. There are some major ideas to keep in mind when selecting the set $\mathcal{P}$. First, the approximate probability density functions should attempt to model the behavior of the system. For example, if the system is known to have limit cycle behavior, the approximate probability density functions $p(x|\theta)$ should model that. The probability density functions should also be chosen based on knowledge obtained from systems which have known solutions to the Fokker-Planck equation. Additionally, the probability density functions should be chosen so that computations can be done efficiently.

In the first method that will be presented, the approximate probability density functions are chosen to be Gaussian probability density functions with unknown variances. Since the true probability density function for the response of linear system is Gaussian, the Gaussian approximation is expected to work well for systems with mild nonlinearities. Additionally, Gaussian probability density functions enable computations to be done efficiently, especially in the case of polynomial non-

linearities, as simple relations exist for the moments of Gaussian probability density functions.

For highly nonlinear systems, Gaussian probability density functions may provide a poor approximation to the true probability density function of the system. Standard Gaussian approximations become especially poor when trying to compute quantities that are sensitive to the tails of the distribution, such as reliability. In these cases, non-Gaussian probability density functions can be chosen to approximate the probability density function for the nonlinear system. In sections 4.3 and 4.4 two different methods are presented for selecting non-Gaussian probability density functions to obtain approximate solutions to the Fokker-Planck equation.

4.1.2 Comments About the Norm

Two main ideas are considered when selecting the norm to be used in (4.4). First, the norm should be chosen so that the cost function

$$J(\theta) \equiv \|L_{nl}(x)p(x|\theta)\| \tag{4.5}$$

is a quantity which, if minimized, should result in a good estimate to the quantities of interest. Additionally, as in the case of selecting the set $\mathcal{P}$, the norm should be chosen so that evaluations of the cost function (4.5) can be efficiently computed.

To illustrate these ideas, consider the Lebesgue norms (Rudin 1987). The $\mathcal{L}^1$ norm, defined by

$$\|L_{nl}(x)p(x|\theta)\|_{\mathcal{L}^1} \equiv \int_{\mathbb{R}^n} | L_{nl}(x)p(x|\theta) | \, dx$$

provides a good measure of the error in the Fokker-Planck equation, but is not particularly easy to work with. Alternatively, the $\mathcal{L}^\infty$ norm

$$\|L_{nl}(x)p(x|\theta)\|_{\mathcal{L}^\infty} \equiv \sup_{x\in\mathbb{R}^n} |L_{nl}(x)p(x|\theta)|$$

is relatively easy to work with, but does not provide a good estimate to the overall

error in the Fokker-Planck equation.

A norm which satisfies both conditions is the $\mathcal{L}^2$ norm defined by

$$\|L_{nl}(x)p(x|\theta)\|^2_{\mathcal{L}^2} \equiv \int_{\mathbb{R}^n} \left(L_{nl}(x)p(x|\theta) \right)^2 \, dx.$$

Weighted $\mathcal{L}^2$ norms can also be defined by

$$\|L_{nl}(x)p(x|\theta)\|^2_{\mathcal{L}^2(\rho)} \equiv \int_{\mathbb{R}^n} \left(L_{nl}(x)p(x|\theta) \right)^2 \rho(x) \, dx$$

where $\rho(x)$ is a weighting function with $\rho(x) > 0 \; \forall x$. The weighting functions will be chosen to provide more emphasis where desired, while not substantially increasing computational difficulty. More comments on the weighting functions will be made later.

4.2 Probabilistic Linearization

The first method developed along these lines (Polidori and Beck 1996) approximates the nonlinear system (4.1) with a linear system whose corresponding probability density function best solves the Fokker-Planck equation associated with the nonlinear system (4.1).

The linear system chosen to approximate (4.1) is

$$dx(t) = A_{eq}x(t) \, dt + B \, dw(t) \tag{4.6}$$

where A_{eq} is a matrix to be determined such that the probability density function for the response of the linear system (4.6) provides the best approximation to the Fokker-Planck equation for the nonlinear system (4.1).

While the matrix A_{eq} can be determined, it is easier to work with the probability density function associated with the linear system (4.6) directly. Since (4.6) is a linear system, the probability density function for the response will be Gaussian.

Therefore, the probability density functions will be chosen to be of the form

$$p_{lin}(x|\theta) = \frac{1}{(2\pi)^{n/2}\sqrt{\det P(\theta)}} \exp\left(-\frac{1}{2}x^T P^{-1}(\theta)x\right) \tag{4.7}$$

where $P = E[xx^T]$ is the covariance matrix, and the parameters θ are the elements of $P(\theta)$ which need to be determined. The set $\mathcal{P}$ is taken as

$$\mathcal{P}_{lin} = \left\{f \in \mathcal{C}^2(\mathbb{R}^n) : f(x) = p_{lin}(x|\theta),\ \theta \in \mathbb{R}^p\right\}.$$

The approximating probability density function is chosen to solve

$$\min_{p\in\mathcal{P}_{lin}} \|L_{nl}(x)p(x)\|$$

or, equivalently

$$\min_{\theta\in\Theta} \|L_{nl}(x)p_{lin}(x|\theta)\|. \tag{4.8}$$

For reasons discussed in section 4.1.2, the norm is taken to be either the standard $\mathcal{L}^2$ norm or a weighted $\mathcal{L}^2$ norm.

4.2.1 Example 1: Linearly Damped Duffing Oscillator

To illustrate the method, a stochastically excited Duffing oscillator is considered. There is a known solution to the stationary Fokker-Planck equation associated with the Duffing oscillator, and the system has been extensively studied in the past by a variety of methods (e.g., Crandall 1963; Caughey 1963; Jahedi and Ahmadi 1983; Redhorse and Spanos 1992; Dunne 1996; Wojtkiewicz et al. 1996). Results of the probabilistic linearization method are compared with those obtained from equivalent linearization and the more computationally expensive sixth-order cumulant neglect closure technique.

Duffing's equation can be nondimensionalized and put in the form (Crandall

1980)

$$\begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} x_2 \\ -\beta x_2 - x_1 - \epsilon x_1^3 \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sqrt{2\beta} \end{pmatrix} dw(t) \tag{4.9}$$

where $\epsilon, \beta > 0$. The stationary Fokker-Planck equation associated with (4.9) is

$$L_{nl}(x)p(x) = -\frac{\partial}{\partial x_1}(x_2 p) + \frac{\partial}{\partial x_2}\left(\left(\beta x_2 + x_1 + \epsilon x_1^3\right) p\right) + \beta \frac{\partial^2 p}{\partial x_2^2} = 0. \tag{4.10}$$

The exact solution to (4.10) is (Caughey and Ma 1982)

$$p(x_1, x_2) = \frac{\sqrt{\epsilon/\pi}}{e^{\frac{1}{8\epsilon}} K_{1/4}(\frac{1}{8\epsilon})} \exp\left(-\frac{x_1^2}{2} - \epsilon\frac{x_1^4}{4} - \frac{x_2^2}{2}\right) \tag{4.11}$$

where $K_{1/4}(\cdot)$ is a modified Bessel function (Abramowitz and Stegun 1964).

As in (4.7), the approximate probability density functions are chosen as

$$p_{lin}(x|\sigma_{x_1}, \sigma_{x_2}) = \frac{1}{2\pi\sigma_{x_1}\sigma_{x_2}} \exp\left(-\frac{x_1^2}{2\sigma_{x_1}^2} - \frac{x_2^2}{2\sigma_{x_2}^2}\right) \tag{4.12}$$

because the cross-variance $E[x_1 x_2] = E[x_1 \dot{x}_1] = 0$ in the stationary case. Since the damping in (4.9) is linear, the exact probability density function for x_2 is Gaussian, and, for the parameters of (4.9), $\sigma_{x_2} = 1$. Therefore, to simplify the algebra, the approximate probability density functions (4.12) are reduced to

$$p_{lin}(x|\sigma) = \frac{1}{2\pi\sigma} \exp\left(-\frac{x_1^2}{2\sigma^2}\right) \exp\left(-\frac{x_2^2}{2}\right).$$

The error in the Fokker-Planck equation is given by

$$\|L_{nl}(x)p_{lin}(x|\sigma)\|_{\mathcal{L}_2} = \left\| \left(\frac{x_1 x_2}{\sigma^2} - x_1 x_2 - \epsilon x_1^3 x_2\right) p_{lin}(x|\sigma) \right\|_{\mathcal{L}^2}.$$

Computation of the norm involves integrating polynomials multiplied by a Gaussian probability density function, i.e., computing moments. Simple analytical relations exist for the moments of a Gaussian probability density function, enabling the norm to be computed analytically. Using Mathematica (Wolfram 1991) to evaluate the

integral gives

$$\|L_{nl}(x)p_{lin}(x|\sigma)\|^2_{\mathcal{L}^2} = \frac{1}{64\pi}\left(4\sigma^{-3} - 8\sigma^{-1} + (4-12\epsilon)\sigma + 12\epsilon\sigma^3 + 15\epsilon^2\sigma^5\right). \tag{4.13}$$

To minimize the error with respect to σ, set

$$\frac{\partial \|L_{nl}(x)p_{lin}(x|\sigma)\|^2_{\mathcal{L}^2}}{\partial\sigma} = -12\sigma^{-4} + 8\sigma^{-2} + (4-12\epsilon) + 36\epsilon\sigma^2 + 75\epsilon^2\sigma^4 = 0.$$

Letting $s = \sigma^2$, the above equation can be written as

$$-12 + 8s + (4-12\epsilon)s^2 + 36\epsilon s^3 + 75\epsilon^2 s^4 = 0. \tag{4.14}$$

It is easily verified that (4.14) has only one positive real root for all $\epsilon \geq 0$ and that this root gives a minimum of $\|L_{nl}(x)p_{lin}(x|\theta)\|$. An analytic expression for this root can be obtained, but, as this is the solution of a quartic equation, the resulting expression is rather long and is not presented here. The results are shown graphically in Figures 4.3 and 4.4, for $E[x_1^2] = \sigma^2$.

An alternative approach to solving the quartic equation (4.14) is to numerically minimize the Fokker-Planck error (4.13) for the desired values of ϵ. For systems with more complicated nonlinearities, this approach is typically much easier than trying to obtain an analytic expression for the optimal variances.

Comparison with Equivalent Linearization

Applying the method of equivalent linearization to the Duffing equation (4.9) gives

$$\begin{aligned}
\beta_{eq} &= \beta \\
\omega^2_{eq} &= 1 + 3\epsilon\sigma^2_{x_1} \\
\sigma^2_{x_1} &= \frac{\beta}{\beta_{eq}\omega^2_{eq}} \\
\sigma^2_{x_2} &= \frac{\beta}{\beta_{eq}}
\end{aligned}$$

which can be combined to give

$$\sigma_{x_1}^2 = \frac{\sqrt{1+12\epsilon}-1}{6\epsilon} \tag{4.15}$$
$$\sigma_{x_2}^2 = 1.$$

Figure 4.1 shows the approximate probability density functions obtained by the probabilistic and equivalent linearization methods compared with the exact solution (4.11) for $\epsilon = 0.3$. The errors in the Fokker-Planck equation for the two approximation methods are shown in Figure 4.2 for ϵ ranging from 0 to 3.

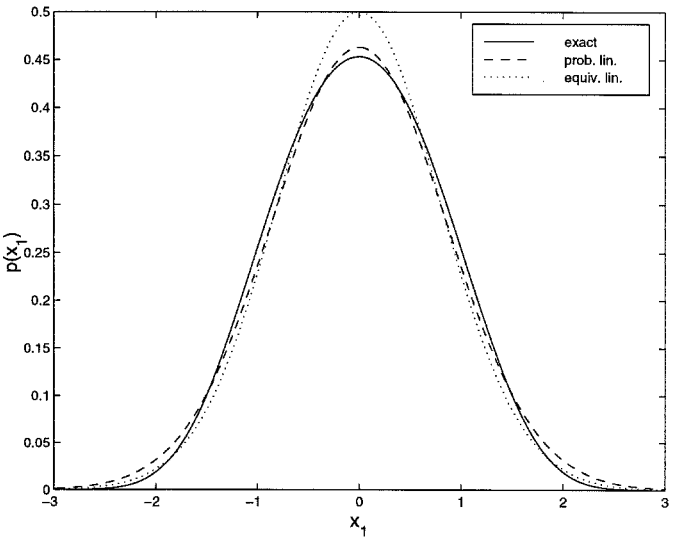


Figure 4.1 Probability density functions for the linearly damped Duffing oscillator, $\epsilon = 0.3$

Mean Square Estimation

As discussed in section 3.2, the mean square values of a process are often of interest. The exact mean square value for the response of the Duffing oscillator (4.9) can be

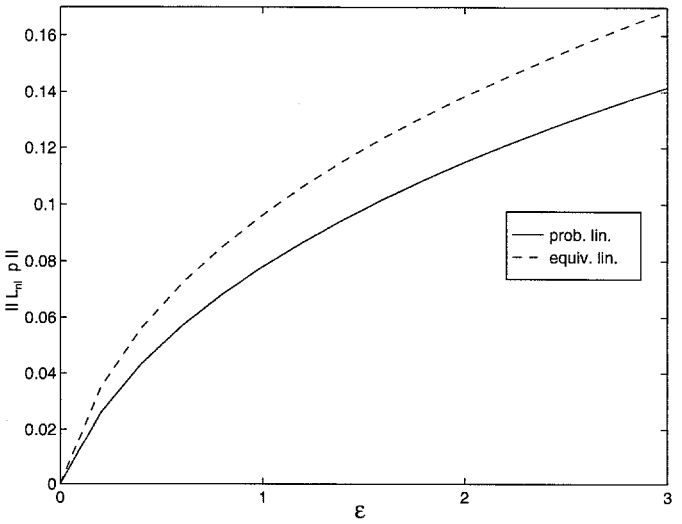


Figure 4.2 Fokker-Planck equation error for the linearly damped Duffing oscillator

obtained from the known probability density function (4.11) as

$$E[x_1^2] = \frac{2\Gamma(\frac{5}{4})\Big(e^{\frac{1}{4\epsilon}}\sqrt{\pi\epsilon}L_{1/4}^{-1/2}(-\frac{1}{4\epsilon}) - {}_1F_1(\frac{5}{4},\frac{3}{2},\frac{1}{4\epsilon})\Big)}{e^{\frac{1}{8\epsilon}}\epsilon^{3/4}K_{1/4}(\frac{1}{8\epsilon})}$$

where $L_{1/4}^{-1/2}(\cdot)$ is a Laguerre polynomial, ${}_1F_1(\cdot)$ is the Kummer confluent hypergeometric function and $\Gamma(\cdot)$ is the Gamma function (Abramowitz and Stegun 1964).

From equation (4.15), the method of equivalent linearization gives the approximation

$$E[x_1^2] = \frac{\sqrt{1+12\epsilon}-1}{6\epsilon}$$

and for the probabilistic linearization method, $E[x_1^2]$ is obtained as the positive, real root of (4.14). As mentioned earlier, the resulting expression is long, and the results are presented graphically.

It was found that the mean square approximations obtained by the method of probabilistic linearization could be improved by using a simple weighting function.

The weighting function $\rho(x) = 1 + x_1^2$ was used in the norm. Although the choice of a weighting function is somewhat arbitrary, this particular function was chosen for the following reasons:

1. To provide more weight to the tails of the distribution in the norm. From Figure 4.1, it can be seen that with no weighting function, a good fit to the actual probability density function is obtained for small values of $|x_1|$ (where $p(x_1)$ is large), while the approximation over-predicts the probability for large values of $|x_1|$ (where $p(x_1)$ is small). Obtaining a good approximation to the probability density function for small values of $|x_1|$ is not as important when trying to estimate the mean square value.

2. The weighting function did not significantly increase any computational effort.

3. It seems reasonable to include a x_1^2 term if trying to approximate $E[x_1^2]$. In this case, the cost function (4.5) is given by

$$J(\theta) = \int_{\mathbb{R}^2} \left(L_{nl}(x)p_{lin}(x|\theta)\right)^2 + x_1^2 \left(L_{nl}(x)p_{lin}(x|\theta)\right)^2 \, dx$$

so that an attempt is made to keep both $L_{nl}p$ and $x_1^2(L_{nl}p)$ "small". This idea is further illustrated later in Figure 4.5 where similar weighting functions are used to approximate the higher moments.

Using this weighting function, the error in the Fokker-Planck equation is computed as

$$\|L_{nl}(x)p_{lin}(x|\sigma)\|_{\mathcal{L}_2(\rho)} = 8\sigma^{-3} - 4\sigma^{-1} - (16 + 24\epsilon)\sigma + (12 - 36\epsilon)\sigma^3 + (60\epsilon + 30\epsilon^2)\sigma^5 + 105\epsilon^2\sigma^7$$

The above expression can be easily minimized with respect to σ.

The mean square values obtained by the various methods are plotted in Figure 4.3 for $0 \leq \epsilon \leq 3$. For larger values of ϵ, the results are plotted in Figure 4.4, along with results obtained by the sixth-order cumulant-neglect closure scheme.

The cumulant-neglect results were obtained by Papadimitriou and Lutes (1996). It is seen that the probabilistic linearization method makes conservative predictions, while the equivalent linearization method under-predicts the response. In addition, it is observed that by using the simple weighting function $\rho = 1+x_1^2$, the mean square values predicted are almost identical to the exact values; the accuracy obtained in this way is even better than the results obtained by the much more computationally expensive sixth-order cumulant-neglect closure method.

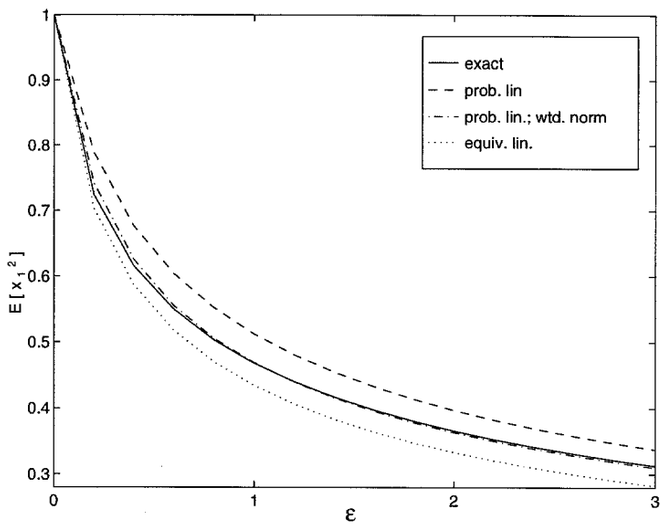


Figure 4.3 $E[x_1^2]$ for the linearly damped Duffing oscillator. $0 \leq \epsilon \leq 3$

The method was also used to try to approximate some of the higher moments for the response of the Duffing oscillator (4.9). Based on the results of the mean square approximation, the weighting function

$$\rho_n(x) = 1 + x_1^n$$

was used in the norm to approximate the n^{th}-order moment, $E[x_1^n]$. Results are shown in Figure 4.5 for the fourth, sixth, and eighth moments. It is seen that with these simple weighting functions a good approximation is obtained to the higher

To illustrate these ideas, consider the linearly damped Duffing oscillator (4.9) with the safe set defined as the region

$$\mathcal{S} = \{x \in \mathbb{R}^2 : x_1 \in (-b, b)\}.$$

The estimates of the stationary outcrossing rates out of the region $\mathcal{S}$ for the linearization methods can be easily computed from the approximate probability density

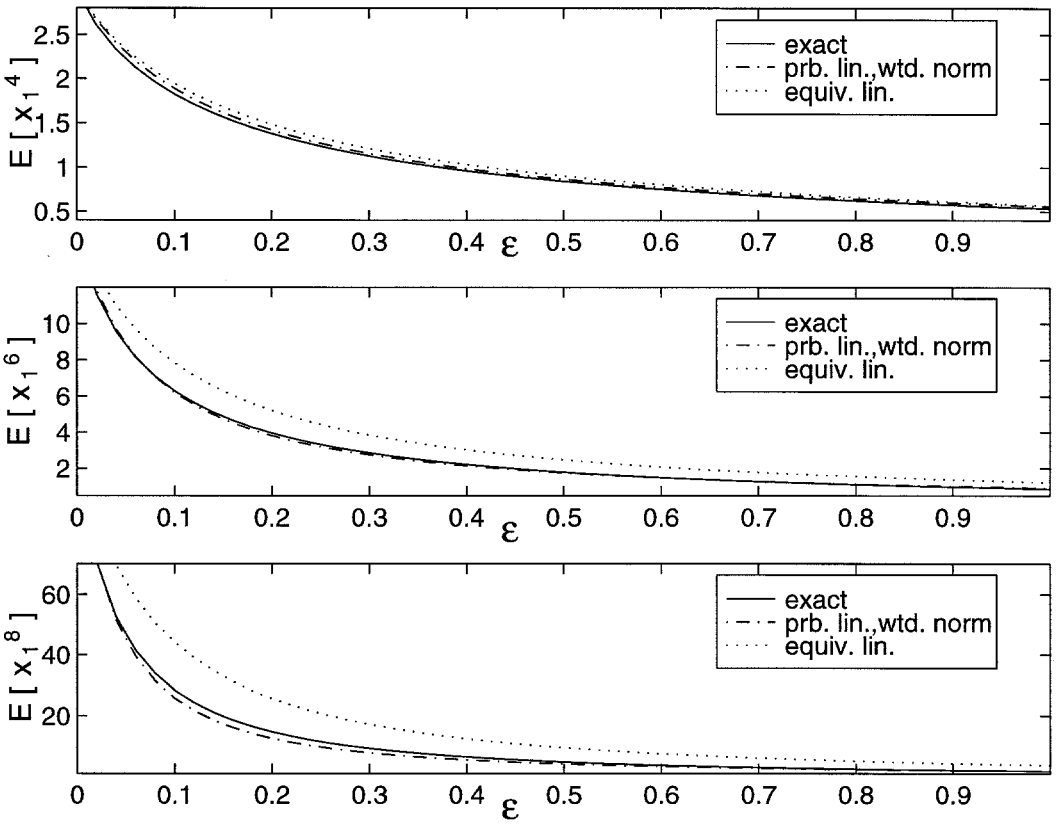


Figure 4.5 Approximations for some of the higher moments of the response for the linearly damped Duffing oscillator. For the probabilistic linearization method, the weighting function $\rho_n(x) = 1 + x_1^n$ was used in the norm.

functions and the formula (3.5), giving

$$\nu = \frac{1}{\pi\sigma_{x_1}} \exp\left(-\frac{b^2}{2\sigma_{x_1}^2}\right). \tag{4.16}$$

The exact value of the outcrossing rate can be computed from the probability density function (4.11) and the formula (3.5) to be

$$\nu_{exact} = \frac{2\sqrt{\epsilon/\pi}}{e^{\frac{1}{8\epsilon}} K_{1/4}(\frac{1}{8\epsilon})} \exp\left(-\frac{b^2}{2} - \epsilon\frac{b^4}{4}\right). \tag{4.17}$$

The values obtained by the linearization methods are compared with the exact value in Figure 4.6. The associated failure probabilities, assuming that the outcrossings are a Poisson process, are shown in Figure 4.7 for the time interval $0 \leq t \leq 20$ when the threshold level is $b = 2.75$.

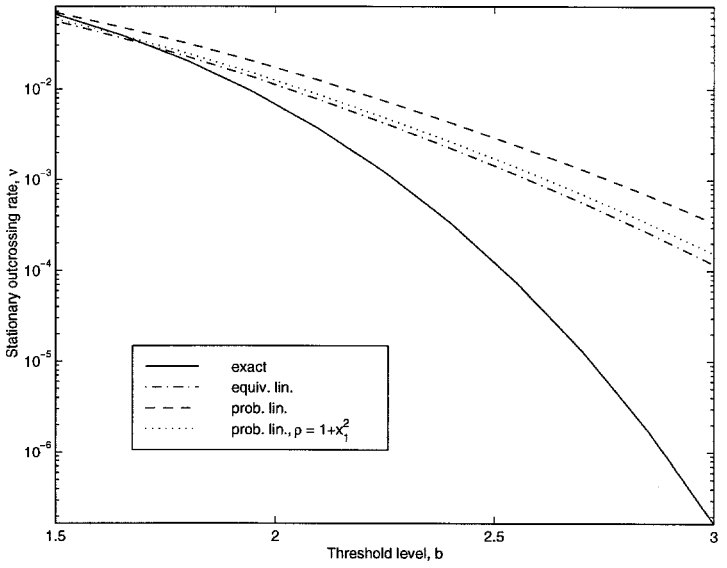


Figure 4.6 Stationary outcrossing rate estimation for the linearly damped Duffing oscillator. $\epsilon = \frac{1}{2}$

Notice that both equivalent and probabilistic linearization drastically over-predict the outcrossing rate and hence the probability of failure. The reason for this is that

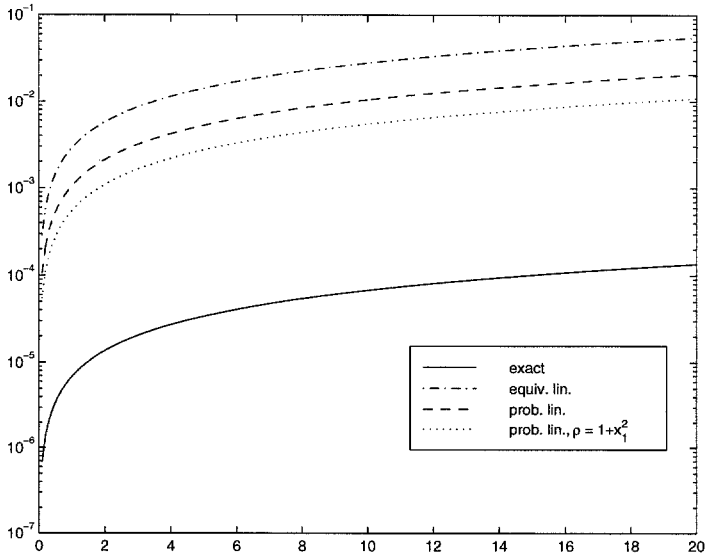


Figure 4.7 Estimation of the failure probability for the linearly damped Duffing oscillator. $\epsilon = \frac{1}{2}$, $b = 2.75$.

the value of the stationary outcrossing rate is highly dependent on the tails of the probability density function, which are known to be non-Gaussian for the nonlinear system. This shortcoming of linearization methods is well known, and a number of equivalent non-linearization methods have been developed to try to improve the results, as discussed in chapter 3.

The probabilistic linearization method can be used to obtain accurate estimates for the outcrossing rate, and hence the reliability, provided that different weighting functions are used in the norm. Recall from Rice's formula (3.5) that for a single degree-of-freedom oscillator, the stationary outcrossing rate of the threshold $x_1 = b$ is given by

$$\nu = \int_0^\infty x_2 \, p(b, x_2) \, dx_2 \tag{4.18}$$

from which it is seen that the stationary outcrossing rate depends only values of the probability density function for $x_1 = b$. Similarly, outcrossing of the threshold

$x_1 = -b$ depends only on values of the probability density function for $x_1 = -b$. For this reason, emphasis should be placed on the regions near $x_1 = b$ and $x_1 = -b$ when minimizing $\|L_{nl}p\|$. To do this, the weighting function

$$\rho(x_1) = \exp\left(-\frac{(x_1 - b)^2}{2\sigma_\rho^2}\right) + \exp\left(-\frac{(x_1 + b)^2}{2\sigma_\rho^2}\right) \tag{4.19}$$

was used in the norm. The variance of the weighting function, σ_ρ^2, is kept small to focus the emphasis near $x_1 = \pm b$. This weighting function is illustrated in Figure 4.8 for the case of $b = 2.75$ and $\sigma_\rho^2 = \frac{1}{8}$.

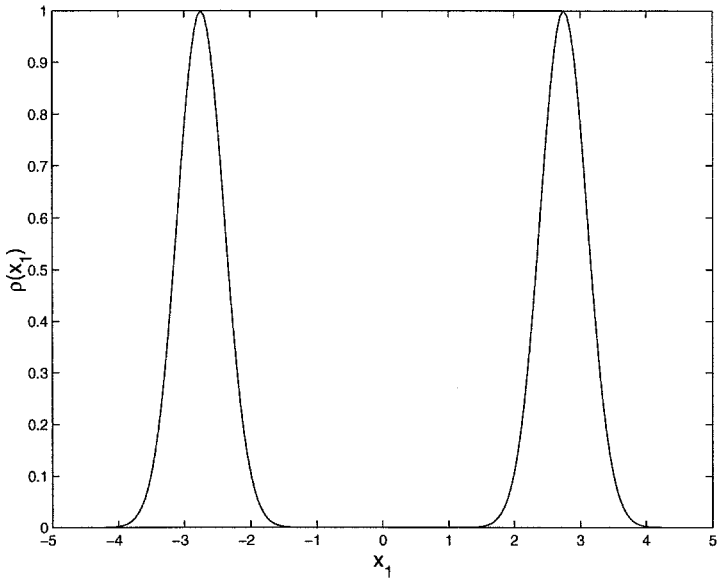


Figure 4.8 Weighting function $\rho(x_1)$ used in the method of probabilistic linearization to estimate the stationary outcrossing rate

As with the weighting functions chosen to estimate the moments, this weighting function was chosen to provide emphasis where desired, while not significantly increasing computational difficulty. It is clear from Figure 4.8 that the weighting function is such that the emphasis is placed on the regions near $x_1 = b$ and $x_1 = -b$. Additionally, since ρ is the sum of two Gaussians, the norm can still be written in

terms of polynomials times Gaussians, and can therefore be evaluated analytically.

With this choice of a weighting function, the method of probabilistic linearization can be easily applied as before. The set of approximate probability density functions is taken to be

$$\mathcal{P}_{lin} = \left\{ p_{lin}(x|\sigma_{x_1}) = \frac{1}{2\pi\sigma_{x_1}} \exp\left(-\frac{x_1^2}{2\sigma_{x_1}^2} - \frac{x_2^2}{2} \right) : \sigma_{x_1} \in \mathbb{R}^+ \right\}$$

and the probabilistic linearization criterion is

$$\min_{\sigma_{x_1} \in \mathbb{R}^+} \| L_{nl}(x) p_{lin}(x|\sigma_{x_1}) \|_{\mathcal{L}^2(\rho)} \tag{4.20}$$

where ρ is as in (4.19). Once the minimization in (4.20) has been performed to give the optimal parameter $\hat{\sigma}_{x_1}$, the estimate to the outcrossing rate is given by (4.16) as

$$\nu \approx \frac{1}{\pi\hat{\sigma}_{x_1}} \exp\left(-\frac{b^2}{2\hat{\sigma}_{x_1}^2} \right). \tag{4.21}$$

Estimates to the outcrossing rate obtained by this method are shown in Figure 4.9 for several values of the variance of the weighting function, σ_ρ^2. As expected, the results improve as σ_ρ^2 decreases and for $\sigma_\rho^2 = \frac{1}{8}$ the results are much better than the equivalent linearization results given in Figure 4.6. However, if σ_ρ^2 is made too small, $\sigma_\rho^2 < \frac{1}{8}$ for this example, good estimates cannot be obtained by this method for large values of b. The reason for this can be seen by examining the minimization criterion in (4.20). The norm is given by

$$\| L_{nl}(x) p_{lin}(x|\sigma_{x_1}) \|_{\mathcal{L}^2(\rho)}^2 = \tag{4.22}$$
$$\frac{1}{2(2\pi)^{3/2}\sigma_{x_1}^2} \int_{-\infty}^{\infty} \left(\frac{x}{\sigma_{x_1}^2} - (x + \epsilon x^3) \right)^2 e^{-\frac{x_1^2}{\sigma_{x_1}^2}} \left(e^{-\frac{(x_1-b)^2}{2\sigma_\rho^2}} + e^{-\frac{(x_1+b)^2}{2\sigma_\rho^2}} \right) dx_1.$$

For most values of σ_ρ^2 the above integral is minimized by selecting σ_{x_1} so that

$$\frac{x_1}{\sigma_{x_1}^2} \approx x_1 + \epsilon x_1^3$$

over the regions near $x_1 = \pm b$. But, as σ_ρ^2 becomes small small and b becomes large, the minimum achieved by having σ_{x_1} satisfy the above condition becomes a local minimum, and a global minimum is achieved at a value of σ_{x_1} very near zero. In this case, the local minimum can still be found by standard minimization techniques and is of use, but the global minimum, which gives a corresponding outcrossing rate approximation very near zero, is of no use. In order to obtain the most accurate estimates for the outcrossing rate, σ_ρ^2 should be kept small so that emphasis is given to regions near $x_1 = \pm b$, but not so small that the optimal value of σ_{x_1} goes to zero. Such a value of σ_ρ^2 can be found by starting with a relatively large value for σ_ρ^2 and reducing σ_ρ^2 until no reasonable results can be obtained.

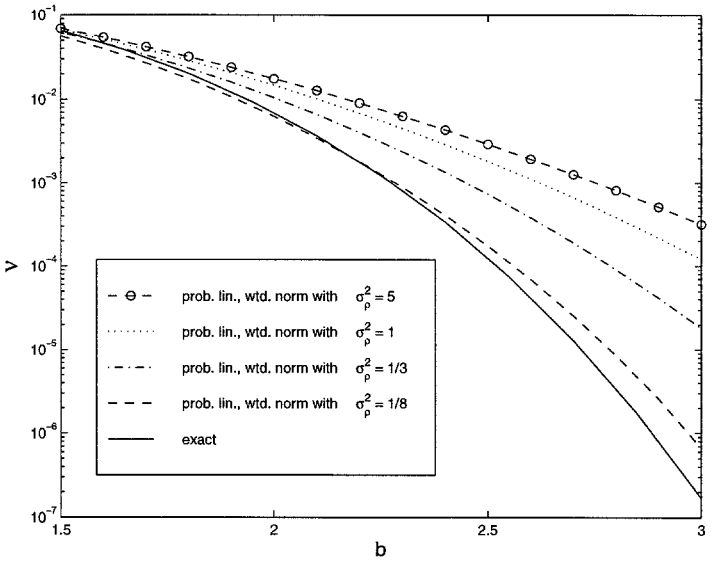


Figure 4.9 Stationary outcrossing rate estimation for the linearly damped Duffing oscillator using weighted norm, where the weighting function is as in (4.19). $\epsilon = \frac{1}{2}$.

4.2.2 Other Interpretations of the Probabilistic Linearization Criterion

In the probabilistic linearization method outlined in section 4.2, a linear system is found whose corresponding probability density function best approximates the Fokker-Planck equation corresponding to the nonlinear equation. Letting $L_{nl}(x)$ and $L_{lin}(x)$ be the forward Kolmogorov operators for the nonlinear system (4.1) and the linear system (4.6), respectively, the associated stationary Fokker-Planck equations are

$$L_{nl}(x)p_{nl}(x) = 0 \tag{4.23}$$

$$L_{lin}(x)p_{lin}(x|\theta) = 0. \tag{4.24}$$

The probabilistic linearization criterion (4.8) can then be written as

$$\min_{\theta\in\Theta} \|L_{nl}(x)p_{lin}(x|\theta) - L_{lin}(x)p_{lin}(x|\theta)\|$$

which is in a form similar to the equivalent linearization criterion (3.9).

Also, by choosing a different norm, this approach can be viewed as minimizing the solution error. Define a norm on $\mathcal{P}$ by

$$\|f\|_{L_{nl}^* L_{nl}} \equiv \int_{\mathbb{R}^n} f(x)L_{nl}^*(x)L_{nl}(x)f(x)\,dx \tag{4.25}$$

where L_{nl}^*, the adjoint of L_{nl}, is the backward Kolmogorov operator defined by (2.23). The norm in (4.25) is just $\mathcal{L}^2$ norm weighted with the operator $L_{nl}^* L_{nl}$. The operator $L_{nl}^* L_{nl}$ is self-adjoint and is positive definite on $\mathcal{P}$ (provided the exact solution to the Fokker-Planck equation is not in $\mathcal{P}$, in which case it is only positive semi-definite), so that (4.25) actually defines a norm.

In this norm, the criterion (4.8) is equivalent to

$$\min_{\theta\in\Theta} \|p_{nl}(x) - p_{lin}(x|\theta)\|_{L_{nl}^* L_{nl}}$$

since

$$\begin{aligned}
\|p_{nl} - p_{lin}\|^2_{L^*_{nl}L_{nl}} &= \langle (p_{nl} - p_{lin}), L^*_{nl}L_{nl}\,(p_{nl} - p_{lin})\rangle_{\mathcal{L}^2} && \text{(eq. 4.25)} \\
&= \langle L_{nl}\,(p_{nl} - p_{lin})\,, L_{nl}\,(p_{nl} - p_{lin})\rangle_{\mathcal{L}^2} && \text{(defn. of } L^*_{nl}\text{)} \\
&= \langle L_{nl}\,p_{lin}, L_{nl}\,p_{lin}\rangle_{\mathcal{L}^2} && \text{(eq. 4.23)} \\
&= \|L_{nl}\,p_{lin}\|^2_{\mathcal{L}^2}
\end{aligned}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{L}^2}$ is the standard inner product on $\mathcal{L}^2$.

This shows that in the norm defined by (4.25), the probabilistic linearization criterion can be viewed as minimizing the difference between the approximate probability density function and the exact (unknown) probability density function.

4.2.3 Comments on Applicability to Multi-Degree-Of-Freedom Systems

Example 1 illustrated the applicability of the method of probabilistic linearization to a single degree-of-freedom oscillator. The method is also applicable to multi-degree-of-freedom systems. As in the case of the single degree-of-freedom oscillator, for polynomial-type nonlinearities, the norms can be evaluated analytically since they will still involve computing moments of a Gaussian process. The problem is then to obtain the elements of the covariance matrix which minimizes the norm, i.e. to solve a p-dimensional minimization problem. The dimension p is the number of elements in the covariance matrix. For a general n-dimensional nonlinear system, this would yield

$$p = \frac{n(n+1)}{2}$$

unknown elements of P, since P is symmetric. In many cases, this number can be reduced since some of the elements of P are usually known; for example, $E[x_i\dot{x}_i] = 0$ for all i. Thus, obtaining an approximate stationary probability density function by the method of probabilistic linearization for an n-dimensional nonlinear system

requires solving a minimization problem in at most $\frac{n(n+1)}{2}$ dimensions.

4.3 Probabilistic Nonlinearization

In the previous section, the probability density function for the nonlinear system of interest was approximated by the probability density function for a linear system. For linear systems, the probability density function is known to be Gaussian, and the Fokker-Planck equation associated with the linear system can easily be solved for the probability density function. Since the response of nonlinear systems is known to be non-Gaussian, it is of interest to see what type of improvement can be obtained by approximating the solution to the Fokker-Planck equation with a non-Gaussian probability density function. In this section, the nonlinear system of interest will be replaced by a different nonlinear system having a known solution to the Fokker-Planck equation. The probability density function for the response of the "simpler" nonlinear system will then be taken as the approximate probability density function for the nonlinear system of interest.

As mentioned earlier, there are very few nonlinear systems in more than two dimensions having known solutions to the stationary Fokker-Planck equation. For this reason, the method presented in this section will be limited to approximating the probability density function for single degree-of-freedom nonlinear oscillators of the form

$$\begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} x_2 \\ -f(x_1, x_2) - g(x_1) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sqrt{2D} \end{pmatrix} dw(t). \tag{4.26}$$

The stationary Fokker-Planck equation associated with (4.26) is

$$L_{nl}\, p = -x_2 \frac{\partial p}{\partial x_1} + g(x_1) \frac{\partial p}{\partial x_2} + \frac{\partial}{\partial x_2} \left(f(x_1, x_2)\, p + D \frac{\partial p}{\partial x_2} \right) = 0. \tag{4.27}$$

As discussed in section 3.1, for nonlinear systems subjected to additive stochastic excitation, many of the known solutions to the Fokker-Planck equation are for single degree-of-freedom oscillators with energy-dependent damping. As in the method of

equivalent nonlinearization (section 3.3.2) the probabilistic nonlinearization method approximates the nonlinear system of interest (4.26) by a system of the form

$$\begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} x_2 \\ -f_{eqnl}(H)\, x_2 - g(x_1) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sqrt{2D} \end{pmatrix} dw(t) \tag{4.28}$$

where H is the Hamiltonian, defined by (3.2). The solution to the stationary Fokker-Planck equation associated with (4.28) is, from (3.3),

$$p_{eqnl}(x_1, x_2) = a \exp\left(-\frac{1}{D} \int_0^{H(x_1,x_2)} f_{eqnl}(\eta)\, d\eta \right).$$

While the equivalent nonlinear function $f_{eqnl}(H)$ can be chosen to be of any form, a simple choice is to choose a power series in H

$$f_{eqnl}(H|\theta) = \sum_{i=1}^{p} \theta_i H^{i-1}$$

for some $p \in \mathbb{Z}^+$, giving

$$p_{eqnl}(x|\theta) = a \exp\left(-\frac{1}{D} \sum_{i=1}^{p} \theta_i \frac{H^i(x)}{i} \right).$$

The number of terms included in the power series is chosen based on the properties of the function $f(x_1, x_2)$. The parameters, θ, of $f_{eqnl}(H|\theta)$ can then be determined so that the probability density function associated with the nonlinear system (4.28) minimizes the Fokker-Planck equation error

$$\min \| L_{nl}(x) p_{eqnl}(x|\theta) \|.$$

As in the probabilistic linearization method, the step of finding the equivalent nonlinear system (4.28) will be bypassed and the probability density functions will be approximated directly. The set of approximate probability density functions is cho-

sen as

$$\mathcal{P}_H = \left\{ p_{eqnl} \in \mathcal{C}^2(\mathbb{R}^2) : p_{eqnl}(x|\theta) = a \exp\left(-\frac{1}{D} \sum_{i=1}^{p} \theta_i \frac{H^i(x)}{i} \right), \theta \in \mathbb{R}^p \right\}. \tag{4.29}$$

Note that given any $p_{eqnl} \in \mathcal{P}_H$, $p_{eqnl}(x|\theta)$ is only a function of $H(x)$ and can be therefore be written in the form $p_{eqnl}(x|\theta) = h(H(x)|\theta)$, giving

$$\begin{aligned} \frac{\partial p_{eqnl}}{\partial x_1} &= \frac{\partial h}{\partial H}\frac{\partial H}{\partial x_1} = g(x_1)\frac{\partial h}{\partial H} \\ \frac{\partial p_{eqnl}}{\partial x_2} &= \frac{\partial h}{\partial H}\frac{\partial H}{\partial x_2} = \quad x_2\frac{\partial h}{\partial H} \end{aligned}$$

so that

$$-x_2\frac{\partial p_{eqnl}}{\partial x_1} + g(x_1)\frac{\partial p_{eqnl}}{\partial x_2} = 0$$

and the error in the stationary Fokker-Planck equation (4.27) is reduced to

$$L_{nl}(x)p_{eqnl}(x|\theta) = \frac{\partial}{\partial x_2}\left(f(x_1, x_2)p_{eqnl}(x|\theta) + D\frac{\partial p_{eqnl}(x|\theta)}{\partial x_2} \right). \tag{4.30}$$

The probabilistic nonlinearization criterion is then

$$\min_{p \in \mathcal{P}_H} \| L_{nl}(x)p(x) \|$$

or,

$$\min_{\theta \in \mathbb{R}^p} \| L_{nl}(x)p_{eqnl}(x|\theta) \|. \tag{4.31}$$

While (4.31) is a valid criterion, finding the optimal parameters would require a significant amount of computation since evaluating the norm requires a two-dimensional numerical integration, and performing the minimization would require computing the norm several times. In the next section, some simplifications to (4.31) will be made which will significantly reduce the computational burden.

4.3.1 Simplifications of the Probabilistic Nonlinearization Criterion

It was mentioned above that finding the optimal parameters for the probabilistic nonlinearization criterion (4.31) requires a number of two-dimensional numerical integrations. To enable computations to be done efficiently, two simplifications will be made.

First, a somewhat simpler expression than $\|L_{nl}p_{eqnl}\|$ is minimized. Substituting (4.30) into (4.31) gives the criterion

$$\min_{\theta\in\mathbb{R}^p} \left\| \frac{\partial}{\partial x_2} \left(f(x_1,x_2)p_{eqnl}(x|\theta) + D\frac{\partial p_{eqnl}(x|\theta)}{\partial x_2} \right) \right\|. \tag{4.32}$$

The first simplification made is that instead of minimizing the derivative of the quantity in parenthesis, the quantity itself is minimized, i.e.

$$\min_{\theta\in\mathbb{R}^p} \left\| f(x_1,x_2)p_{eqnl}(x|\theta) + D\,\frac{\partial p_{eqnl}(x|\theta)}{\partial x_2} \right\|. \tag{4.33}$$

The derivative of p_{eqnl} with respect to x_2 can easily be evaluated, giving

$$\frac{\partial p_{eqnl}(x|\theta)}{\partial x_2} = -\frac{1}{D}\left(\sum_{i=1}^{p}\theta_i H^{i-1}\right) x_2\, p_{eqnl}(x|\theta).$$

Substituting into (4.33) gives

$$\min_{\theta\in\mathbb{R}^p} \left\| \left(f(x_1,x_2) - x_2\sum_{i=1}^{p}\theta_i H^{i-1} \right) p_{eqnl}(x|\theta) \right\|,$$

which, using the standard $\mathcal{L}^2$ norm, is

$$\min_{\theta\in\mathbb{R}^p} \int_{\mathbb{R}^2} \left(f(x_1,x_2) - x_2\sum_{i=1}^{p}\theta_i H^{i-1} \right)^2 p_{eqnl}^2(x|\theta)\,dx. \tag{4.34}$$

Notice that this criterion is virtually identical to the criterion used by the method of equivalent nonlinearization (section 3.3.2); the equivalent nonlinearization criterion (3.22) is the same as (4.34) with p_{eqnl}^2 replaced by p_{eqnl}. While this first simplification

has simplified the form of the criterion from (4.32) to (4.34), the resulting integral still cannot be evaluated analytically for most systems.

The second simplification comes in evaluating the integral in (4.34). As it is written, evaluating the integral requires a two-dimensional numerical integration. To allow analytical computation of the integral, $p_{eqnl}(x|\theta)$ is replaced in the integral by $p_{lin}(x_1, x_2)$ obtained by the method of probabilistic linearization. This is equivalent to using the weighting function

$$\rho(x) = \frac{p_{lin}^2(x)}{p_{eqnl}^2(x)}$$

in the norm. The probabilistic linearization method can be very efficiently applied, and making this substitution will enable the optimal parameters to be computed very efficiently.

Thus, after the two simplifications, the probabilistic nonlinearization criterion becomes

$$\min_{\theta \in \mathbb{R}^p} \int_{\mathbb{R}^2} \left(f(x_1, x_2) - x_2 \sum_{i=1}^{p} \theta_i H^{i-1} \right)^2 p_{lin}^2(x_1, x_2)\, dx. \tag{4.35}$$

Now, (4.35) can be easily minimized with respect to θ. To find the minimum, set

$$\frac{\partial}{\partial \theta_k} \left\{ \int_{\mathbb{R}^2} \left(f(x_1, x_2) - x_2 \sum_{i=1}^{p} \theta_i H^{i-1} \right)^2 p_{lin}^2(x_1, x_2)\, dx \right\} = 0 \qquad k = 1, 2, \ldots, p.$$

Evaluating the derivatives gives a linear equation for the optimal parameters $\hat{\theta}$

$$A\hat{\theta} = b. \tag{4.36}$$

The matrix $A \in \mathbb{R}^{p \times p}$ and vector $b \in \mathbb{R}^p$ are given by

$$\begin{aligned} A_{ij} &= \langle H^{i-1} x_2, H^{j-1} x_2 \rangle \\ b_i &= \langle f, H^{i-1} x_2 \rangle \end{aligned}$$

where the inner product is defined by

$$\langle \phi, \psi \rangle \equiv \int_{\mathbb{R}^2} \phi(x)\psi(x)p_{lin}^2(x)\,dx. \tag{4.37}$$

With the optimal parameters known, the approximate probability density function is then given by

$$p_{eqnl}(x) = a \exp\left(-\frac{1}{D}\sum_{i=1}^{p} \hat{\theta}_i \frac{H^i(x)}{i}\right). \tag{4.38}$$

A chart outlining the steps for the method of probabilistic nonlinearization is given in Figure 4.10. Notice that before obtaining the approximation $p_{eqnl}(x)$, the probabilistic linearization approximation $p_{lin}(x)$ is obtained. The relative accuracy of the two approximations can be compared by computing $\|L_{nl}(x)p_{lin}(x)\|$ and $\|L_{nl}(x)p_{eqnl}(x)\|$.

4.3.2 Example 2: Nonlinearly Damped Duffing Oscillator

To illustrate the method, a nonlinearly damped Duffing oscillator is considered. This oscillator has been previously studied by Elishakoff and Cai (1993).

$$\begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} x_2 \\ -\beta x_2 - \alpha x_2^3 - \gamma x_1 - \epsilon x_1^3 \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sqrt{2D} \end{pmatrix} dw(t) \tag{4.39}$$

In the notation of (4.26),

$$\begin{aligned} f(x_2) &= \beta x_2 + \alpha x_2^3 \\ g(x_1) &= \gamma x_1 + \epsilon x_1^3 \end{aligned}$$

giving

$$H(x_1, x_2) = \frac{x_2^2}{2} + \gamma \frac{x_1^2}{2} + \epsilon \frac{x_1^4}{4}.$$

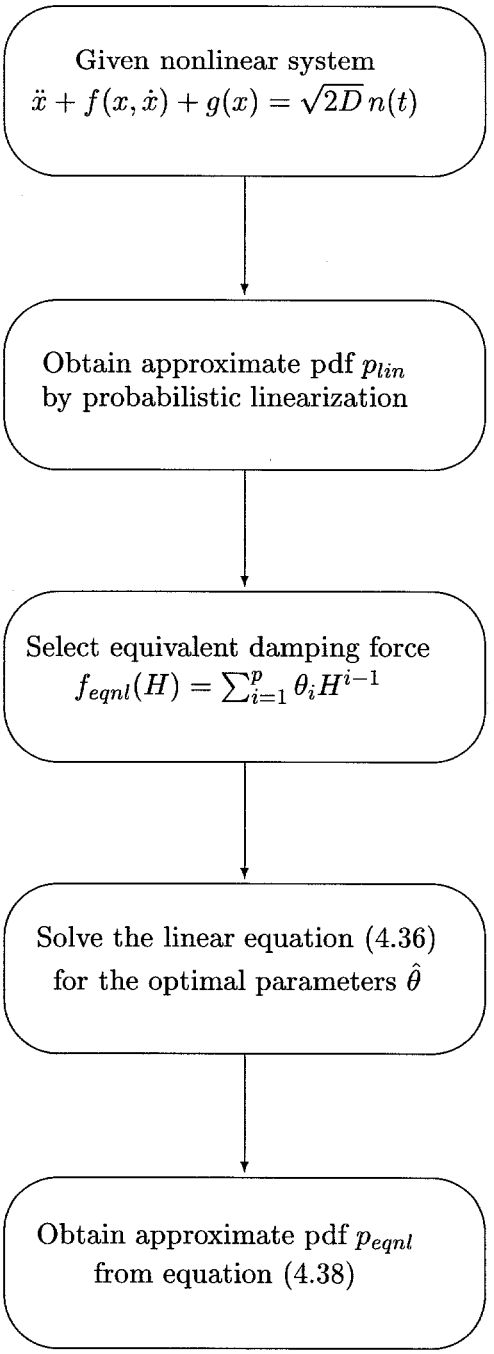


Figure 4.10 Flow chart illustrating steps in probabilistic nonlinearization method

As in (4.29), the equivalent system is chosen by replacing $f(x_1, x_2)$ with $f_{eqnl}(H|\theta)x_2$ where $f_{eqnl}(H|\theta)$ is a power series in H. Since

$$f(x_2) = \left(\beta + \alpha x_2^2\right) x_2$$

it seems reasonable to approximate this function by the following power series

$$f_{eqnl}(H|\theta) = \theta_1 + \theta_2 H.$$

The series is truncated at order H so that the highest power of x_2 in the equivalent damping term $f_{eqnl}(H)x_2$ is the same as the highest power in the original system $f(x_2)$.

The set of approximate probability density functions is then

$$\mathcal{P}_H = \left\{ p \in \mathcal{C}^2(\mathbb{R}^2) : p(x) = a \exp\left(-\frac{1}{D}\left(\theta_1 H(x) + \theta_2 \frac{H^2(x)}{2} \right) \right), \theta \in \mathbb{R}^2 \right\}.$$

The set of linear equations (4.36) for the optimal parameters is given by

$$\begin{pmatrix} \langle x_2, x_2 \rangle & \langle x_2, Hx_2 \rangle \\ \langle x_2, Hx_2 \rangle & \langle Hx_2, Hx_2 \rangle \end{pmatrix} \begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{pmatrix} = \begin{pmatrix} \langle f(x_2), x_2) \rangle \\ \langle f(x_2), Hx_2 \rangle \end{pmatrix} \tag{4.40}$$

where the inner product is as in (4.37).

All of the quantities in (4.40) involving $\langle \cdot, \cdot \rangle$ can be evaluated once p_{lin} has been computed by the probabilistic linearization method. With these quantities known, (4.40) is a linear system of equations in the two unknowns $\hat{\theta}_1$ and $\hat{\theta}_2$, which can be easily solved. Once $\hat{\theta}_1$ and $\hat{\theta}_2$ are known, the approximate probability density function is given by

$$p_{eqnl}(x|\hat{\theta}) = a \exp\left(-\frac{\hat{\theta}_1}{D} H(x) - \frac{\hat{\theta}_2}{2D} H^2(x) \right).$$

All of the integrals up to this point can be done analytically. Evaluating the normalization constant a and other quantities such as moments and expected outcrossing

rates requires numerical integration.

In order to compare this method with some of the other methods presented earlier, two different cases are considered. First, all of the parameters except for the coefficient ϵ of the nonlinear stiffness term are fixed, and the approximate probability density function are computed for various values of ϵ. The resulting mean square values are shown in 4.11 and the errors in the Fokker-Planck equation are shown in Figure 4.13. In the other case, all of the parameters except for the coefficient α of the nonlinear damping term are kept fixed, and the approximate probability density functions are obtained for various values of α. The mean square values are shown in Figure 4.12 and the Fokker-Planck equation error in Figure 4.14. In all cases, it is seen that the probabilistic nonlinearization provides more accurate results than the linearization or partial linearization results. The simulation results presented in Figures 4.11 and 4.12 are from Elishakoff and Cai (1993).

The approximate probability density functions obtained by the methods are also plotted. Figures 4.15–4.18 show the probability density functions as both surface plots and contour plots. The marginal probability density functions for x_1 and x_2 defined by

$$\begin{aligned} p(x_1) &= \int_{-\infty}^{\infty} p(x_1, x_2)\, dx_2 \\ p(x_2) &= \int_{-\infty}^{\infty} p(x_1, x_2)\, dx_1 \end{aligned}$$

are shown in Figures 4.19 and 4.20. In these plots, the marginal probability density functions for various approximate methods are compared with histograms of the response obtained by numerical simulation of the system; 400,000 points were used to make the histograms. Although the probabilistic nonlinearization method gives good approximations to the mean square values and provides a good approximation to the probability density function $p(x_2)$, it is seen that the probability density function $p(x_1)$ obtained by the method of probabilistic nonlinearization decays much more rapidly as $|x_1|$ increases than the simulation results or the other approximate methods. It will be seen in section 4.4.3 that this leads to large errors when ap-

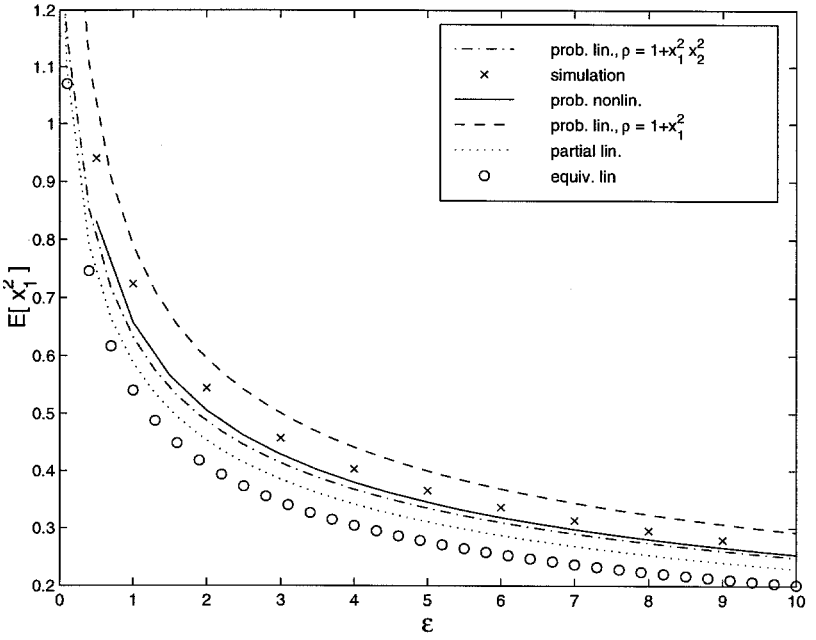


Figure 4.11 Mean square values as a function of the stiffness nonlinearity for the nonlinearly damped Duffing oscillator (4.39). $D = \pi, \alpha = 0.5, \beta = 0.1, \gamma = 1$.

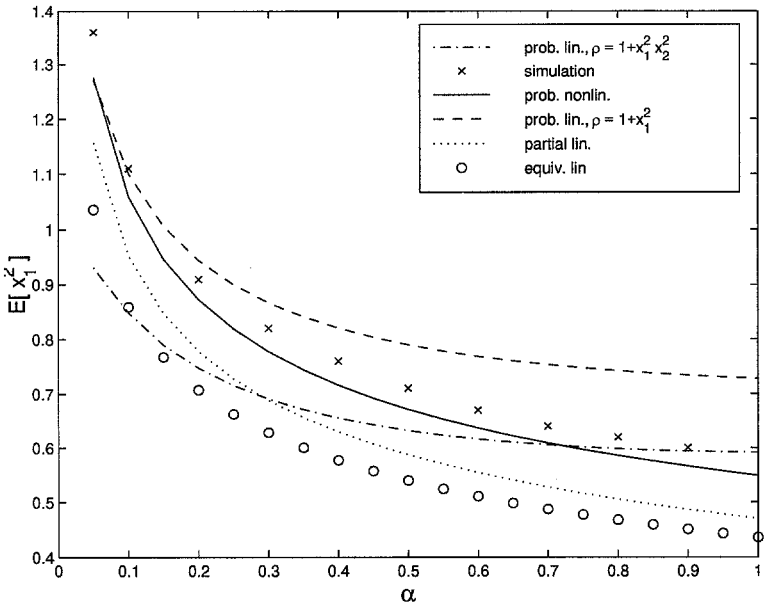


Figure 4.12 Mean square values as a function of the damping nonlinearity for the nonlinearly damped Duffing oscillator (4.39). $D = \pi, \beta = 0.1, \epsilon = 1, \gamma = 1$

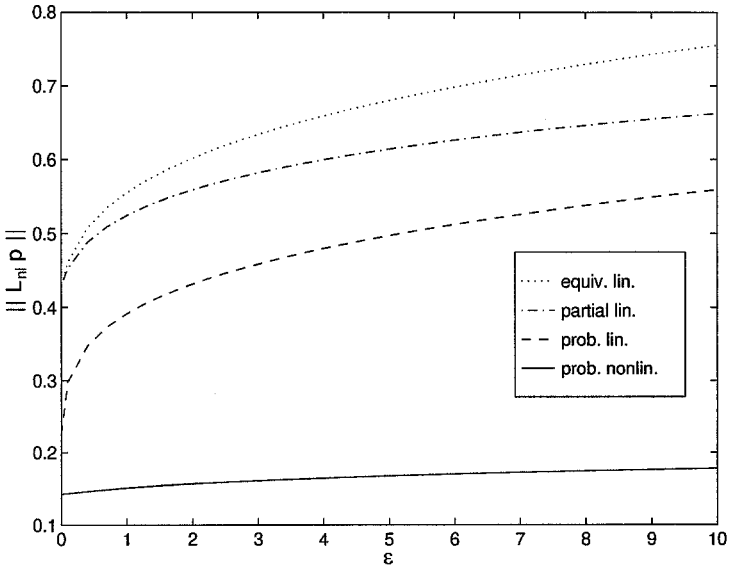


Figure 4.13 Error in the Fokker-Planck equation as a function of stiffness nonlinearity for the nonlinearly damped Duffing oscillator (4.39). $D = \pi, \alpha = 0.5, \beta = 0.1, \gamma = 1$.

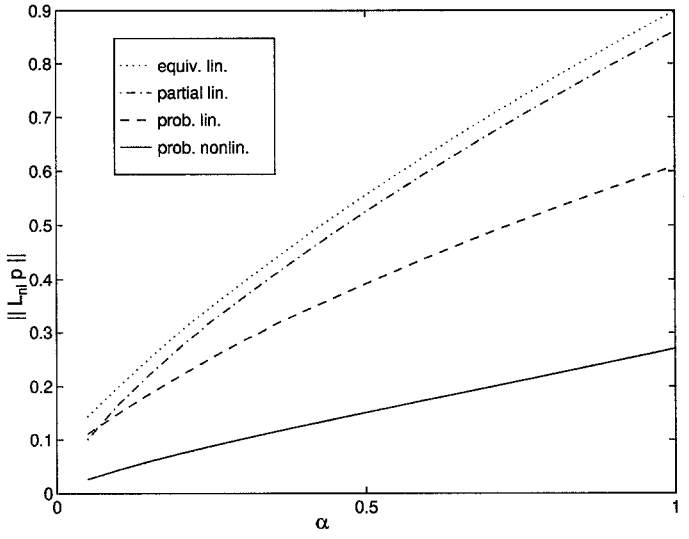


Figure 4.14 Error in the Fokker-Planck equation as a function of damping nonlinearity for the nonlinearly damped Duffing oscillator (4.39). $D = \pi, \beta = 0.1, \gamma = 1, \epsilon = 1$.

proximating the outcrossing rate for this system. Further results for this example, including outcrossing rate estimates, will be given in section 4.4.3.

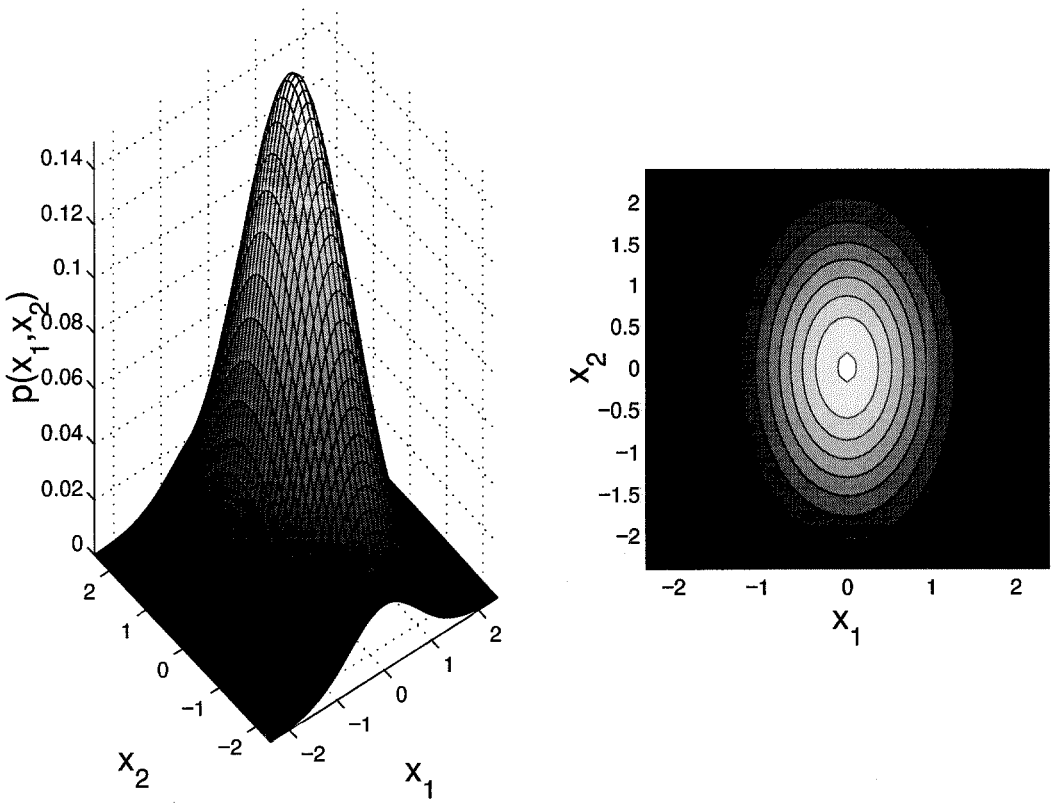


Figure 4.15 Stationary probability density function approximation obtained by the method of equivalent linearization for the nonlinearly damped Duffing oscillator. $\alpha = 0.5, \beta = 0.1, \gamma = 1, \epsilon = 1$.

4.4 Direct Approximation of the Probability Density Function

The previous methods presented approximated the nonlinear system of interest with a different system for which the corresponding Fokker-Planck equation function can be solved. As discussed earlier, there are relatively few systems which have a known

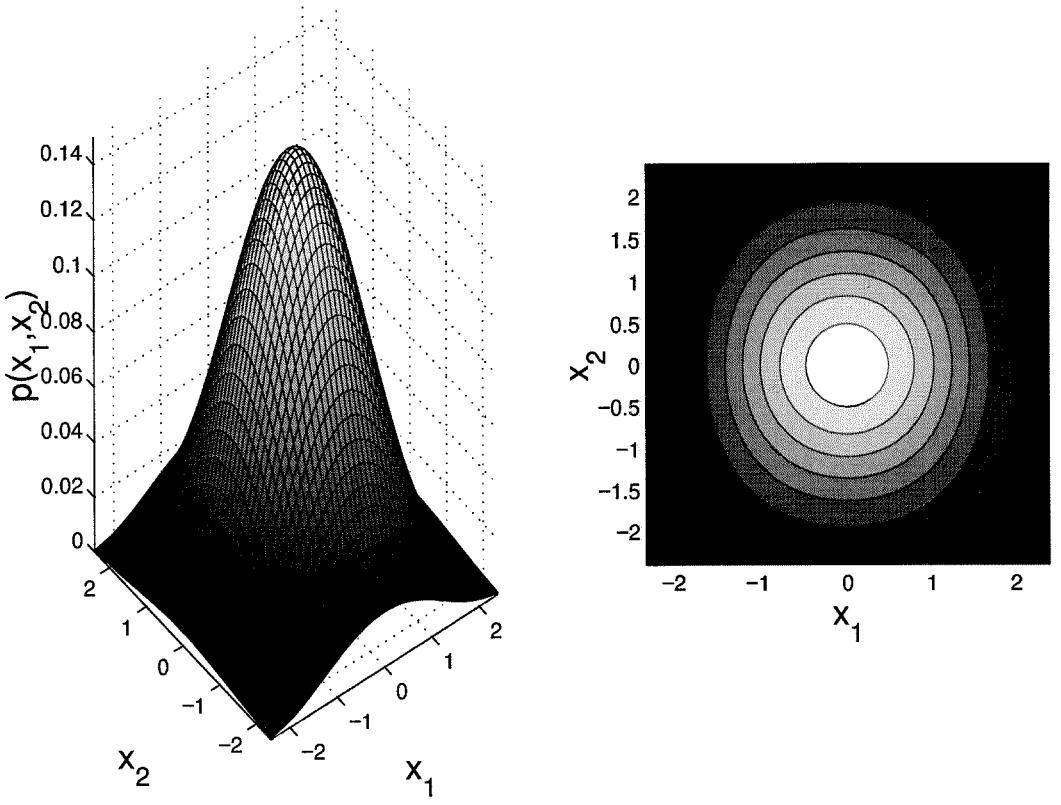


Figure 4.16 Stationary probability density function approximation obtained by the method of partial linearization for the nonlinearly damped Duffing oscillator. $\alpha = 0.5, \beta = 0.1, \gamma = 1, \epsilon = 1$

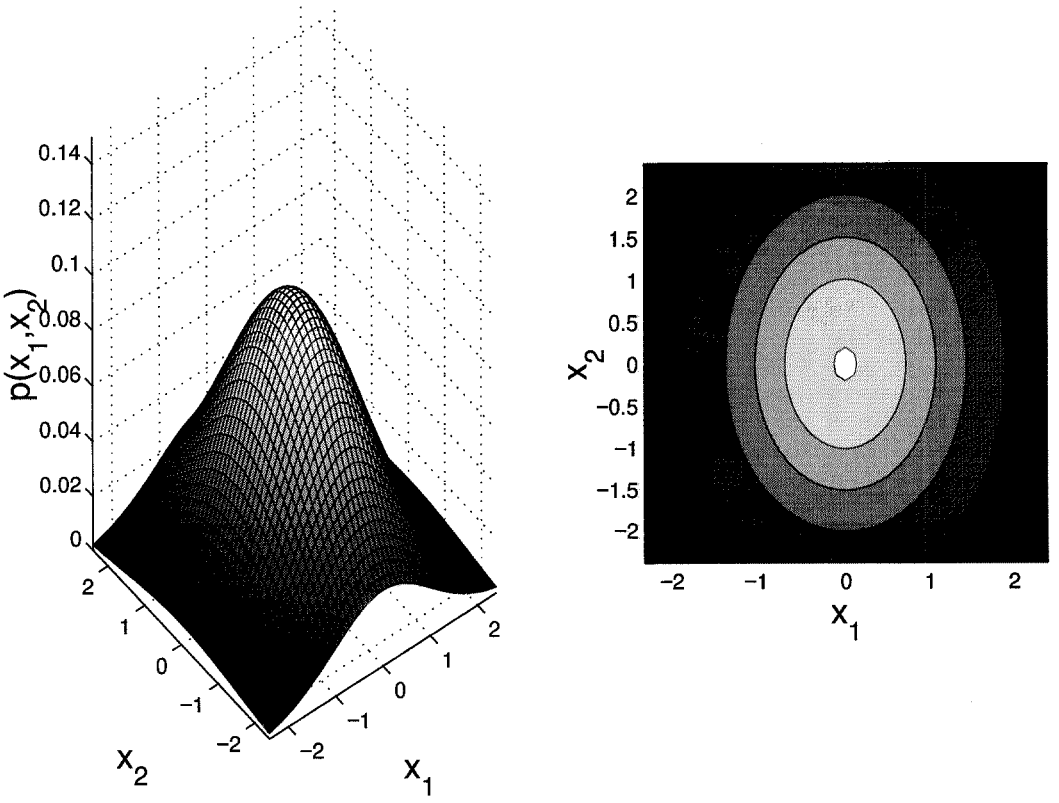


Figure 4.17 Stationary probability density function approximation obtained by the method of probabilistic linearization for the nonlinearly damped Duffing oscillator. $\alpha = 0.5, \beta = 0.1, \gamma = 1, \epsilon = 1$.

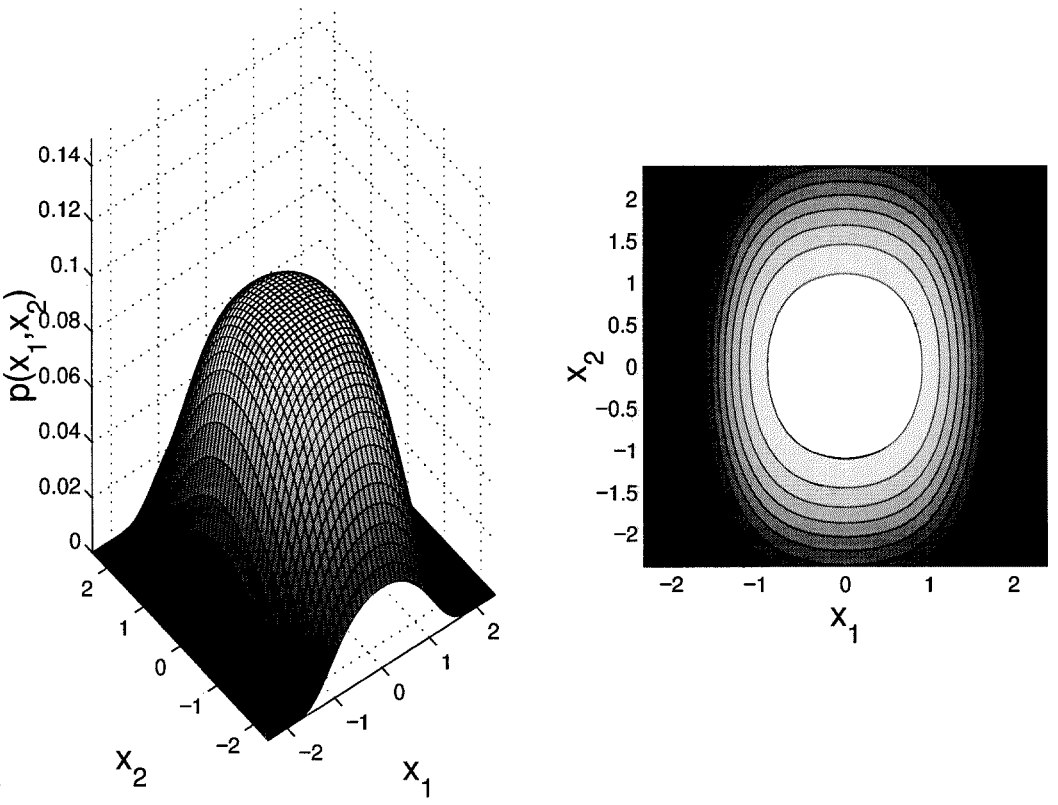


Figure 4.18 Stationary probability density function approximation obtained by the method of probabilistic nonlinearization for the nonlinearly damped Duffing oscillator. $\alpha = 0.5, \beta = 0.1, \gamma = 1, \epsilon = 1$

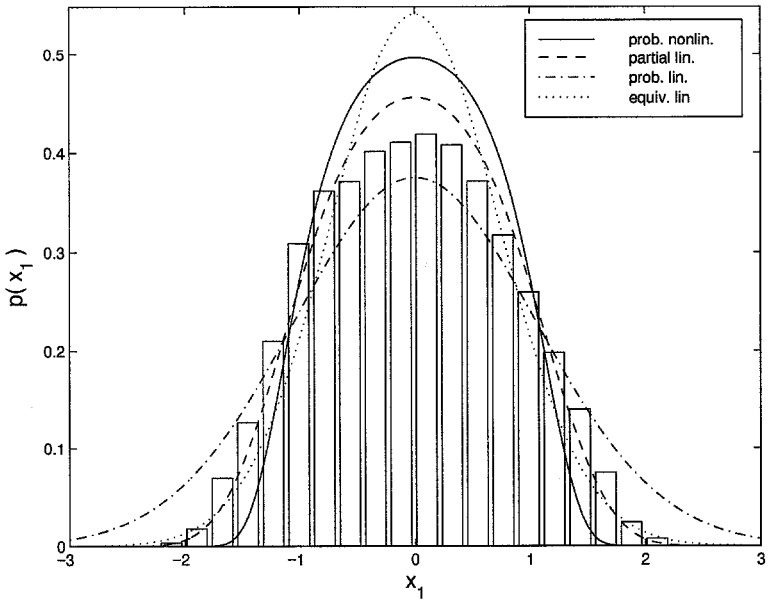


Figure 4.19 Probability density function $p(x_1)$ for the nonlinearly damped Duffing oscillator for the various approximation methods. The histogram shows results from numerical simulation of the system.

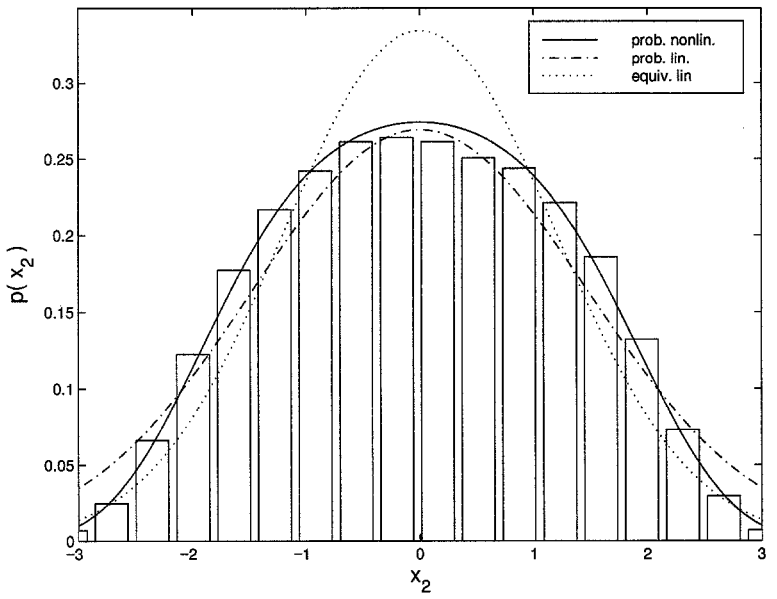


Figure 4.20 Probability density function $p(x_2)$ for the nonlinearly damped Duffing oscillator for the various approximation methods. The methods of equivalent and partial linearization give the same approximation to $p(x_2)$. The histogram shows results from numerical simulation of the system.

solution to the Fokker-Planck equation, and a given nonlinear system may not be well approximated by any of the solvable nonlinear systems. In this case, the set of approximate probability density functions can be chosen to be of a different form, where the form is chosen based on knowledge of the system.

In this section, another method is presented for obtaining non-Gaussian probability density functions to approximate the solutions to the stationary Fokker-Planck equation for a given nonlinear system. Unlike section 4.3, the approximate probability density functions are not chosen to correspond to solutions of "equivalent" systems with energy-dependent damping. In fact, the approximate probability density functions chosen are such that no equivalent system

$$\begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} x_2 \\ -f_{equiv}(x_1, x_2) - g_{equiv}(x_1) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sqrt{2D} \end{pmatrix} dw(t)$$

may exist whose stationary probability density function is of the form of the approximate probability density functions.

4.4.1 Selecting the Approximate Probability Density Functions

This section presents a method for selecting the set of approximate probability density functions for single degree-of-freedom systems. The approximate probability density functions are chosen with two objectives in mind. First, the approximate solutions are chosen so that some of the terms in the Fokker-Planck equation error are identically zero. Additionally, the form of the approximate solutions is chosen so that if the given system is one of the systems having a known solution to the Fokker-Planck equation, the approximate method will actually give the exact solution.

For a given nonlinear dynamical system

$$\begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ -f(x_1, x_2) - g(x_1) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sqrt{2D} \end{pmatrix} dw(t)$$

the associated stationary Fokker-Planck equation is

$$-\frac{\partial\,(x_2 p)}{\partial x_1}+\frac{\partial\,(fp)}{\partial x_2}+\frac{\partial\,(gp)}{\partial x_2}+D\frac{\partial^2 p}{\partial x_2^2}=0$$

which can be rearranged as

$$-x_2\frac{\partial p}{\partial x_1}+g(x_1)\frac{\partial p}{\partial x_2}+\frac{\partial}{\partial x_2}\left(f(x_1,x_2)p+D\frac{\partial p}{\partial x_2}\right)=0.$$

The term in parenthesis will be identically zero if

$$\frac{\partial p}{\partial x_2}=-\frac{1}{D}f(x_1,x_2)p$$

which is accomplished by choosing

$$p_F(x_1,x_2|\theta)=a\exp\left(-\frac{\theta}{D}Q(x_1)\right)\exp\left(-\frac{1}{D}F(x_1,x_2)\right) \tag{4.41}$$

where $Q(x_1)$ is an arbitrary function of x_1, θ is a parameter to be determined, a is a normalization constant and

$$F(x_1,x_2)=\int_0^{x_2} f(x_1,\xi)\,d\xi.$$

The form of the function $Q(x_1)$ is chosen based on systems having known solutions to the Fokker-Planck equation. A desirable property for an approximation method is to be able to give the exact solution for systems having known solutions. An attempt is made to choose $Q(x_1)$ so that the approximate method will actually yield the exact solution for systems with known solutions to the Fokker-Planck equation. With this in mind, one of the simplest choices for $Q(x_1)$ is to choose

$$Q(x_1)=G(x_1)=\int_0^{x_1} g(\xi)\,d\xi \tag{4.42}$$

which would yield the exact solution in the case of linear damping. This is the choice considered in the examples. A slightly more complicated choice for $Q(x_1)$

can be chosen which will yield the exact solution for any system having energy-dependent damping, as explained in appendix A. For the examples considered in this work, in which $f(x_1, x_2)$ is independent of x_1, this alternative choice also gives $Q(x_1) = G(x_1)$.

With $Q(x_1)$ given by (4.42), the set of approximate probability density functions is given by

$$\mathcal{P}_F = \left\{ p_F \in \mathcal{C}^2(\mathbb{R}^2) : p_F(x|\theta) = a \exp\left(-\frac{\theta}{D} G(x_1) - \frac{1}{D} F(x_1, x_2) \right), \theta > 0 \right\}$$

and the criterion for obtaining the best approximation is given as

$$\min_{p_F \in \mathcal{P}_F} \| L_{nl}(x) p_F(x|\theta) \|. \tag{4.43}$$

As in the case of probabilistic nonlinearization, this provides a valid criterion for measuring the error, but can be computationally expensive to work with. For typical systems, computing the norm requires numerical integration, and to find the optimal parameters, the integration needs to be performed many times.

To simplify the criterion, recall that the set $\mathcal{P}_F$ was chosen so that

$$f(x_1, x_2) p_F + D \frac{\partial p_F}{\partial x_2} = 0 \qquad \forall p_F \in \mathcal{P}_F.$$

Therefore, the Fokker-Planck equation error reduces to

$$\begin{aligned} L_{nl} p_F &= g(x_1) \frac{\partial p_F}{\partial x_2} - x_2 \frac{\partial p_F}{\partial x_1} \\ &= \frac{1}{D} \left(\left(\theta g(x_1) + \frac{\partial F}{\partial x_1} \right) x_2 - g(x_1) f(x_1, x_2) \right) p_F(x|\theta). \end{aligned}$$

Part of the difficulty in minimizing the norm of the above expression is that the normalization constant for the probability density function $p_F(x|\theta)$ is a complicated function of θ. To simplify the computations, the following criterion is adopted

$$\min_{\theta} E \left[\left(\left(\theta g(x_1) + \frac{\partial F}{\partial x_1} \right) x_2 - g(x_1) f(x_1, x_2) \right)^2 \right]. \tag{4.44}$$

Differentiating with respect to θ to obtain the minimum gives

$$\theta = \frac{E[g^2(x_1)\, f(x_1, x_2)\, x_2 - g(x_1)\, x_2 \frac{\partial F}{\partial x_1}]}{E[g^2(x_1)\, x_2^2]}. \tag{4.45}$$

Although this criterion seems different than the other methods presented in this chapter, it can be thought of as minimizing the weighted $\mathcal{L}^2$ norm of the Fokker-Planck equation error where the weighting function is $1/p_F(x|\theta)$. An interesting relationship between the criterion (4.45) and those suggested by the methods of equivalent and partial linearization is presented in section 4.5.

Notice that if $f(x_1, x_2)$ is independent of x_1, the criterion reduces to

$$\theta = \frac{E[\, x_2\, f(x_2)\,]}{E[\, x_2^2\,]} \tag{4.46}$$

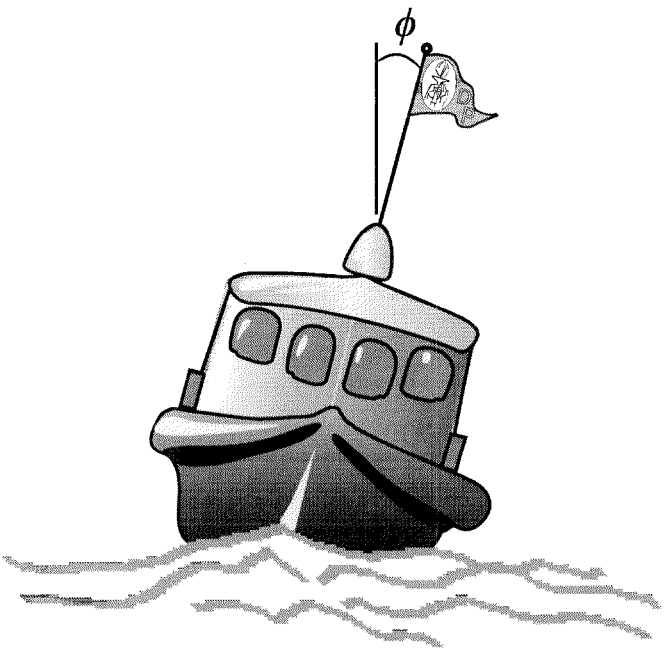
which is the same as the criterion proposed in the methods of equivalent linearization (3.10a) and partial linearization (3.20). The only difference in the criteria is that different probability density functions are used to evaluate the expectations.

4.4.2 Example 3: Rolling Ship

For small angles, the equation of motion for a rolling ship is given by (Roberts 1982; Gawthrop et al. 1988)

$$\ddot{\phi} + \left(\beta + n_1|\dot{\phi}|\right)\dot{\phi} + \left(\omega^2 + n_2\phi^2\right)\phi = \frac{1}{I_{roll}} M(t)$$

where ϕ is the angle from the vertical, ω is the undamped natural frequency for roll, β is the linear damping ratio, n_1 and n_2 are nonlinear damping and stiffness coefficients, respectively, I_{roll} is the effective roll inertia of the ship, and $M(t)$ is the applied moment from the waves. The equation contains a quadratic damping term commonly found in applications involving fluid flow or fluid dampers as well as a



cubic hardening term. Modeling $M(t)$ as Gaussian white noise leads to

$$\begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} x_2 \\ -\left(\omega^2 + n_2 x_1^2\right) x_1 - \left(\beta + n_1 |x_2|\right) x_2 \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sqrt{2D} \end{pmatrix} dw(t) \tag{4.47}$$

where $x_1 = \phi$, $x_2 = \dot{\phi}$ and D is the amplitude of the excitation.

For the system (4.47),

$$\begin{aligned} f(x_2) &= \beta x_2 + n_1 |x_2| x_2 \\ g(x_1) &= \omega^2 x_1 + n_2 x_1^3 \end{aligned}$$

giving

$$\begin{aligned} F(x_2) &= \beta \frac{x_2^2}{2} + n_1 \frac{x_2^2 |x_2|}{3} \\ G(x_1) &= \omega^2 \frac{x_1^2}{2} + n_2 \frac{x_1^4}{4}. \end{aligned}$$

The approximate probability density function is taken as

$$p_F(x) = a \exp\left(-\frac{\theta}{2D}G(x_1) - \frac{1}{D}F(x_2)\right)$$

where, from (4.46),

$$\theta = \frac{E[\,x_2\, f(x_2)\,]}{E[\,x_2^2\,]}. \tag{4.48}$$

The effects of the cubic stiffness nonlinearity have already been investigated, and in to obtain some simple analytical results for the nonlinear damping term, the first case considered is $\beta = n_2 = 0$ and $\omega = 1$ so that the system reduces to the quadratically damped oscillator

$$\begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} x_2 \\ -n_1 x_2|x_2| - x_1 \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sqrt{2D} \end{pmatrix} dw(t). \tag{4.49}$$

The more general case will be studied in section 5.2.2. For the quadratically damped oscillator (4.49), the approximate probability density function is

$$p_F(x) = a \exp\left(-\frac{\theta x_1^2}{2D} - \frac{n_1 x_2^2|x_2|}{3D}\right) \tag{4.50}$$

and, from (4.48), θ is given by

$$\theta = n_1 \frac{E[\,x_2^2\,|x_2|\,]}{E[\,x_2^2\,]} = \left(\frac{n_1}{3D}\right)^{\frac{2}{3}} \Gamma\left(\tfrac{1}{3}\right) \tag{4.51}$$

where $\Gamma(\cdot)$ is the Gamma function.

The mean square values can easily be determined from the probability density function

$$E[x_1^2] = E[x_2^2] = \left(\frac{3D}{n_1}\right)^{\frac{2}{3}} \frac{1}{\Gamma(\frac{1}{3})} = 0.776 \left(\frac{D}{n_1}\right)^{\frac{2}{3}}.$$

The error in the Fokker-Planck equation can also be computed analytically; in

the case of $D = 1$, the error is

$$\|L_{nl}p_F\|_{\mathcal{L}^2} = 0.0625 n_1^{\frac{1}{3}}.$$

Comparison with Equivalent Linearization

For the parameter values considered in (4.49), the method of equivalent linearization gives

$$\begin{aligned} \omega_{eq}^2 &= 1 \\ \beta_{eq} &= n_1 \frac{E[x_2^2|x_2|]}{E[x_2^2]}. \end{aligned} \tag{4.52}$$

Equation (4.52) is the same as the previous formula for θ (4.51). The resulting values will be different though, since the expectations are taken with respect to different probability density functions.
Evaluating (4.52) gives

$$\beta_{eq} = 2n_1\sqrt{\frac{2}{\pi}}\,\sigma_{x_2}$$

from which the mean square values are computed

$$E[x_1^2] = E[x_2^2] = \left(\frac{D}{2n_1}\sqrt{\frac{\pi}{2}}\right)^{\frac{2}{3}} = 0.732\left(\frac{D}{n_1}\right)^{\frac{2}{3}}.$$

The error in the Fokker-Planck equation can also be calculated; for $D = 1$, the error is given by

$$\|L_{nl}p_{eqlin}\| = 0.28 n_1.$$

Notice that the mean square values for the method of equivalent linearization are about 6% lower than those predicted by the non-Gaussian approximate probability density function for all values of the nonlinearity parameter, n_1. However, the error in the Fokker-Planck equation grows much faster in n_1 for the Gaussian

approximation than for the non-Gaussian approximation.

Comparison with Equivalent Nonlinearization

The quadratically damped oscillator is one of the systems for which the integrals arising in the equivalent nonlinearization method can be analytically evaluated, and the results can be presented in a simple form. The solution presented here is taken from Roberts and Spanos (1990). The nonlinear damping term $f(x_2) = n_1|x_2|x_2$ is replaced with an "equivalent" nonlinear damping given by

$$f_{eqnl}(x) = \alpha\sqrt{H}x_2$$

where $H = \frac{1}{2}\left(x_1^2 + x_2^2\right)$ and α is chosen to solve

$$\min_{\alpha} E\left[\left(\alpha\sqrt{H}x_2 - n_1|x_2|x_2\right)^2\right].$$

In Roberts and Spanos (1990), the optimal value for α is determined and the corresponding probability density function for the equivalent nonlinear system is given by

$$p_{eqnl}(x) = \frac{3}{2\pi\Gamma(\frac{2}{3})}\left(\frac{8n_1}{9\pi}\right)^{2/3}\exp\left(-\frac{8n_1}{9\pi}(2H(x))^{3/2}\right). \tag{4.53}$$

The mean square values can be computed from (4.53) and are given by

$$E[x_1^2] = E[x_2^2] = 0.765\left(\frac{D}{n_1}\right)^{\frac{2}{3}}.$$

which are a little more than 1% lower than those predicted by the probability density function in equation (4.50). The mean square values for the various methods are plotted in Figure 4.22 along with values obtained by numerical simulation.

The error in the Fokker-Planck equation is computed by numerical integration and is plotted in Figure 4.21. While the equivalent nonlinearization method provides a better fit than equivalent linearization, the error still grows much faster in n_1 for

the equivalent nonlinearization method than the new method.

The marginal probability density functions $p(x_1)$ and $p(x_2)$ for the various methods are shown in Figures 4.23 and 4.24, along with histograms obtained from numerical simulation of the system. Eight hundred thousand data points were used to make the histograms. From both figures, it is seen that the non-Gaussian probability density functions provide a better fit to the simulated data than the equivalent linearization results, and it appears as though the best fit to the simulated data is obtained by using the direct approximation of the probability density function.

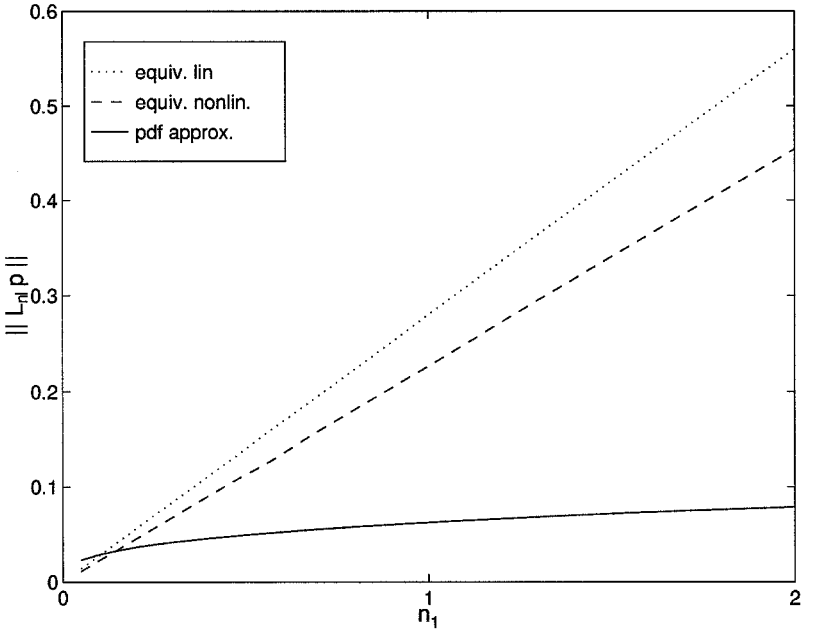


Figure 4.21 Fokker-Planck equation error for the quadratically damped oscillator with $D = 1$.

Reliability Estimates for the Various Methods

The mean square values and Fokker-Planck equation errors for the various methods were presented in Figures 4.21 and 4.22. It was seen that all of the methods gave quite good accuracy for the mean square values for all values of n_1, but that the

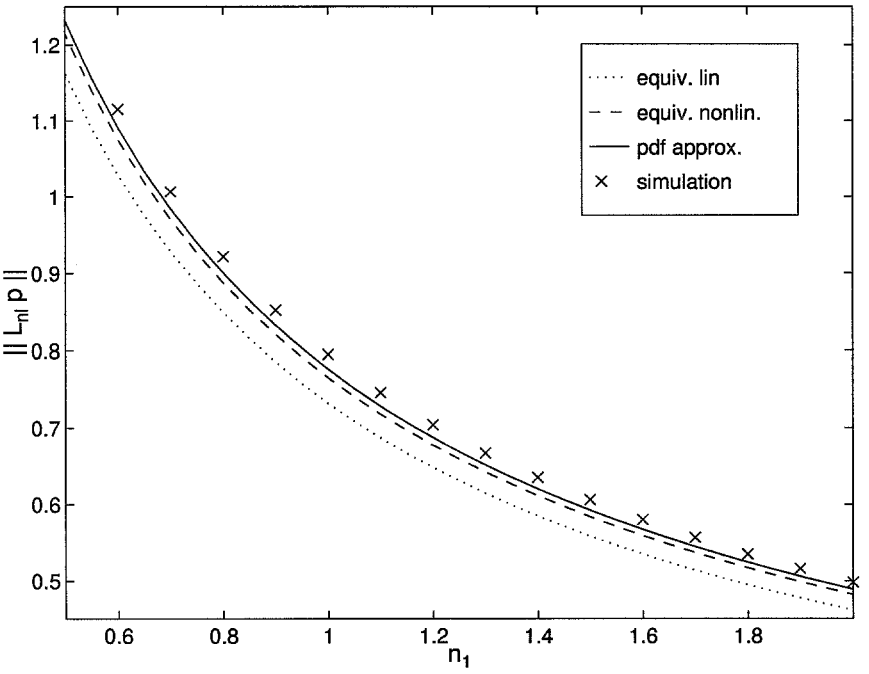


Figure 4.22 Mean square values for the quadratically damped oscillator with $D = 1$.

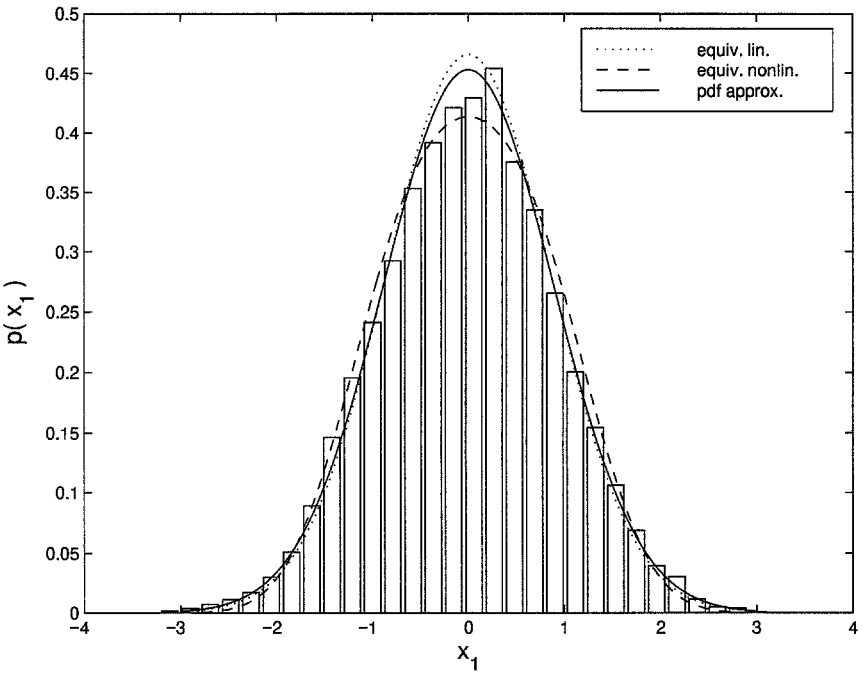


Figure 4.23 Probability density function $p(x_1)$ for the quadratically damped oscillator, $D = 1, n_1 = 1$. The histograms shows results from numerical simulation.

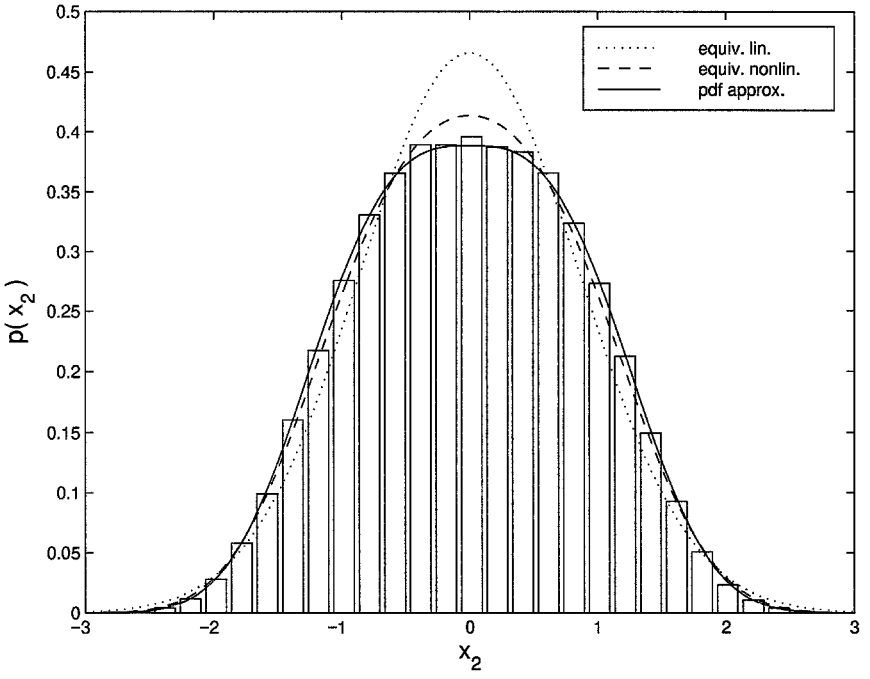


Figure 4.24 Probability density function $p(x_2)$ for the quadratically damped oscillator, $D = 1$, $n_1 = 1$. The histogram shows results from numerical simulation.

error in the Fokker-Planck equation was much smaller for the probability density function p_F than for the equivalent linearization or nonlinearization methods. In this section, the outcrossing rates for the various methods are compared.

In addition to the previously described methods, results are presented from a weighted norm minimization. In this case, the probability density function is chosen as

$$p(x|\theta) = a \exp\left(-\frac{\theta}{D}\frac{x_1^2}{2}\right) \exp\left(-\frac{n_1}{D}\frac{x_2^2|x_2|}{3}\right)$$

as in (4.50), but rather than choosing θ according to (4.46), θ is chosen to minimize the weighted $\mathcal{L}^2$ norm of the Fokker-Planck equation error. As in the reliability estimation for probabilistic linearization, the weighting function is chosen as

$$\rho(x_1) = \exp\left(-\frac{(x_1-b)^2}{2\sigma_\rho^2}\right) + \exp\left(-\frac{(x_1+b)^2}{2\sigma_\rho^2}\right).$$

As in example 1, the value of σ_ρ was reduced until the results suddenly became unreasonable, corresponding to the optimal value of θ going to ∞. The smallest value of σ_ρ which could be used in this example is $\sigma_\rho^2 = \frac{1}{2}$.

The results for the various methods are shown in Figure 4.25, along with results obtained from numerical simulation of the system. It is seen that the equivalent nonlinearization method predicts the outcrossing rate to be much lower than what is predicted by the other methods. The reason for this is that the outcrossing rate decays as $e^{-c_1 b^3}$ for some constant c_1 for the equivalent nonlinearization method whereas the outcrossing rate for the other methods decay as $e^{-c_2 b^2}$.

Note that these results illustrate an important point about equivalent nonlinearization methods. The results obtained by equivalent nonlinearization may be less accurate than those obtained by equivalent linearization if the approximate nonlinear system has sufficiently different behavior than the original nonlinear system. This can be seen in Figure 4.25, where the method of equivalent linearization gives considerably better estimates to the outcrossing rates than those obtained by equivalent nonlinearization for this system.

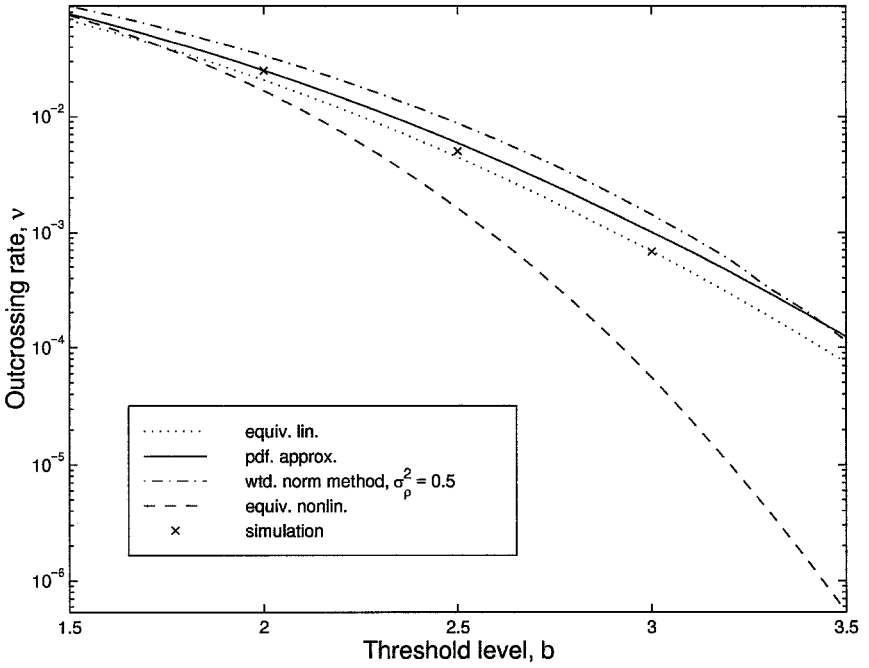


Figure 4.25 Expected outcrossing rates for the quadratically damped oscillator. $n_1 = 1, D = 1$.

4.4.3 Application of the Method to Example 2

It was seen in section 4.3 that the probability density function obtained by the method of probabilistic nonlinearization provided a considerably better fit to the stationary Fokker-Planck equation for the nonlinearly damped Duffing oscillator than either the method of equivalent linearization or probabilistic linearization. It is interesting to see how the results obtained by probabilistic nonlinearization compare with the results obtained by approximation the probability density function as in section 4.4.1.

For the nonlinearly damped Duffing oscillator given by (4.39),

$$\begin{aligned} f(x_2) &= \beta x_2 + \alpha x_2^3 \\ g(x_1) &= \gamma x_1 + \epsilon x_1^3 \end{aligned}$$

giving

$$\begin{aligned} F(x_2) &= \beta\frac{x_2^2}{2} + \alpha\frac{x_2^4}{4} \\ G(x_1) &= \gamma\frac{x_1^2}{2} + \epsilon\frac{x_1^4}{4}. \end{aligned}$$

The probability density function is given by

$$p_F(x|\theta) = a\exp\left(-\frac{\theta}{D}G(x_1)\right)\exp\left(-\frac{1}{D}F(x_2)\right) \tag{4.54}$$

where, from (4.46),

$$\theta = \frac{E[\,x_2\,f(x_2)\,]}{E[\,x_2^2\,]} = \beta + \alpha\frac{E[\,x_2^4\,]}{E[\,x_2^2\,]}$$

which can be evaluated as

$$\theta = \frac{8\alpha D K_{\frac{1}{4}}\left(\frac{\beta^2}{8\alpha D}\right) + \beta^2\left(K_{\frac{3}{4}}\left(\frac{\beta^2}{8\alpha D}\right) - K_{\frac{5}{4}}\left(\frac{\beta^2}{8\alpha D}\right)\right)}{2\beta\left(K_{\frac{3}{4}}\left(\frac{\beta^2}{8\alpha D}\right) - K_{\frac{1}{4}}\left(\frac{\beta^2}{8\alpha D}\right)\right)}$$

where $K_n(\cdot)$ are modified Bessel functions. The Fokker-Planck equation error,

$\|L_{nl}(x)p(x|\theta)\|$ is compared with the error for the methods presented earlier in Figures 4.26 and 4.27. It is seen that the direct approximation of the probability density function provides a better fit to the Fokker-Planck equation than any of the other methods. The mean square values obtained by the various methods are shown in Figures 4.28 and 4.29. From these plots, it is seen that both of the nonlinearization methods do considerably better than the linearization methods. The stationary probability density function p_F is shown in Figure 4.30 and can be compared to those obtained from the various other methods, which are shown in Figures 4.15–4.18. The marginal probability density functions, $p(x_1)$ and $p(x_2)$, are shown in Figures 4.31 and 4.32. The approximate probability density functions are compared with results from numerical simulation, as in Figures 4.19 and 4.20.

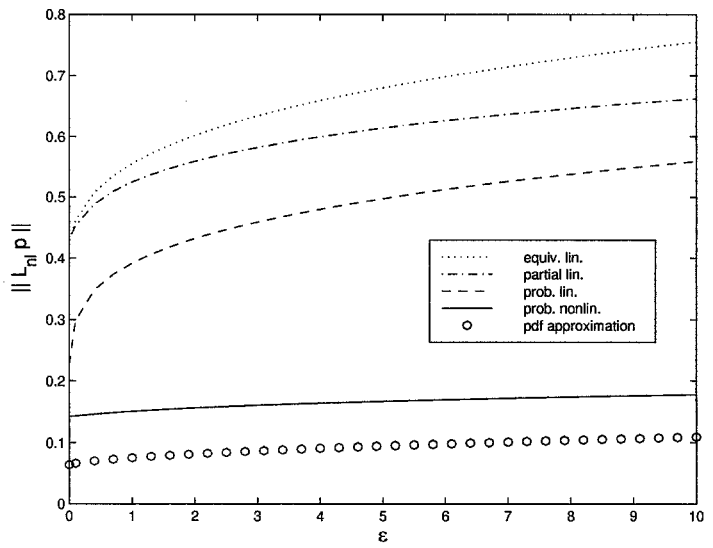


Figure 4.26 Error in the Fokker-Planck equation as a function of stiffness nonlinearity for the nonlinearly damped Duffing oscillator (4.39). $D = \pi, \alpha = 0.5, \beta = 0.1, \gamma = 1$

The most striking difference between the method of probabilistic nonlinearization and the direct probability density function approximation method appears when

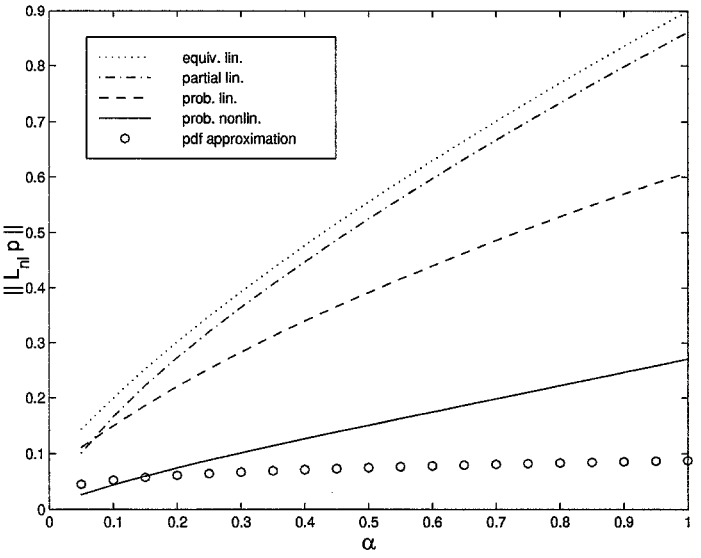


Figure 4.27 Error in the Fokker-Planck equation as a function of damping nonlinearity for the nonlinearly damped Duffing oscillator (4.39). $D = \pi, \beta = 0.1, \gamma = 1, \epsilon = 1$

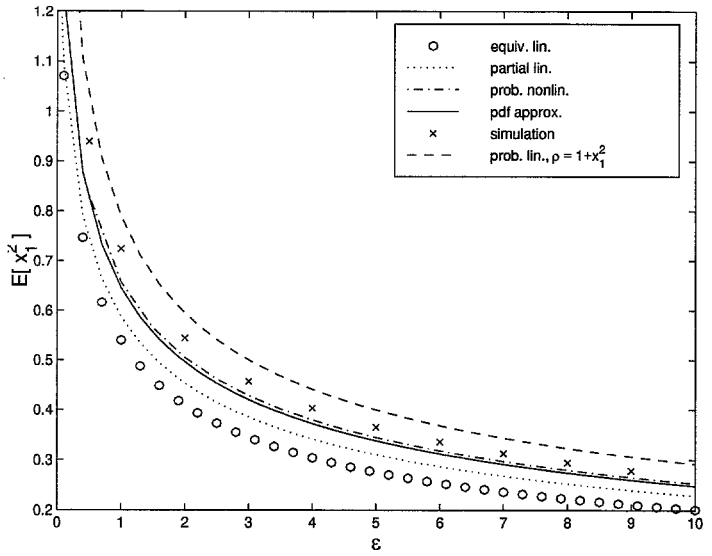


Figure 4.28 Mean square values as a function of the stiffness nonlinearity for the nonlinearly damped Duffing oscillator (4.39). $D = \pi, \alpha = 0.5, \beta = 0.1, \gamma = 1$

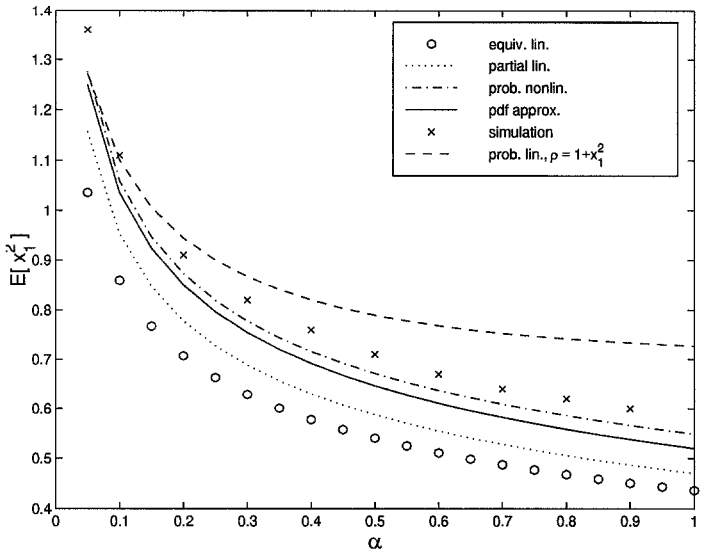


Figure 4.29 Mean square values as a function of the damping nonlinearity for the nonlinearly damped Duffing oscillator (4.39). $D = \pi, \beta = 0.1, \epsilon = 1, \gamma = 1$

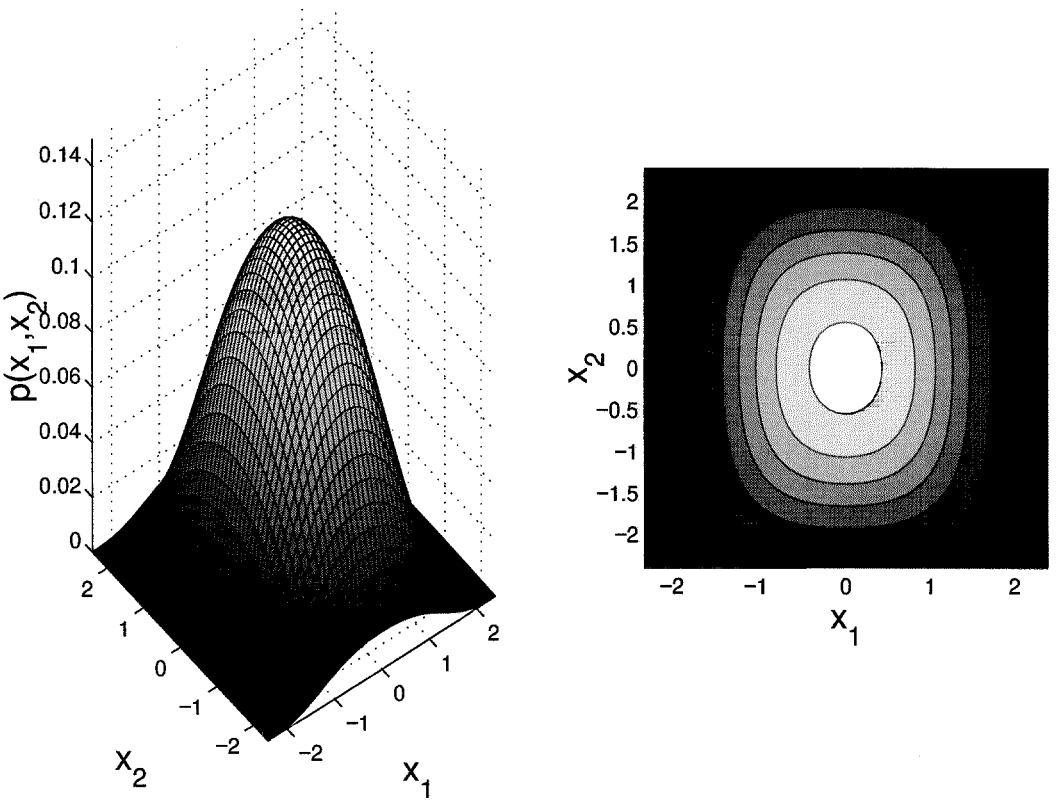


Figure 4.30 Stationary probability density function p_F obtained by the directly approximating the Fokker-Planck equation for the nonlinearly damped Duffing oscillator. $\alpha = 0.5, \beta = 0.1, \gamma = 1, \epsilon = 1$

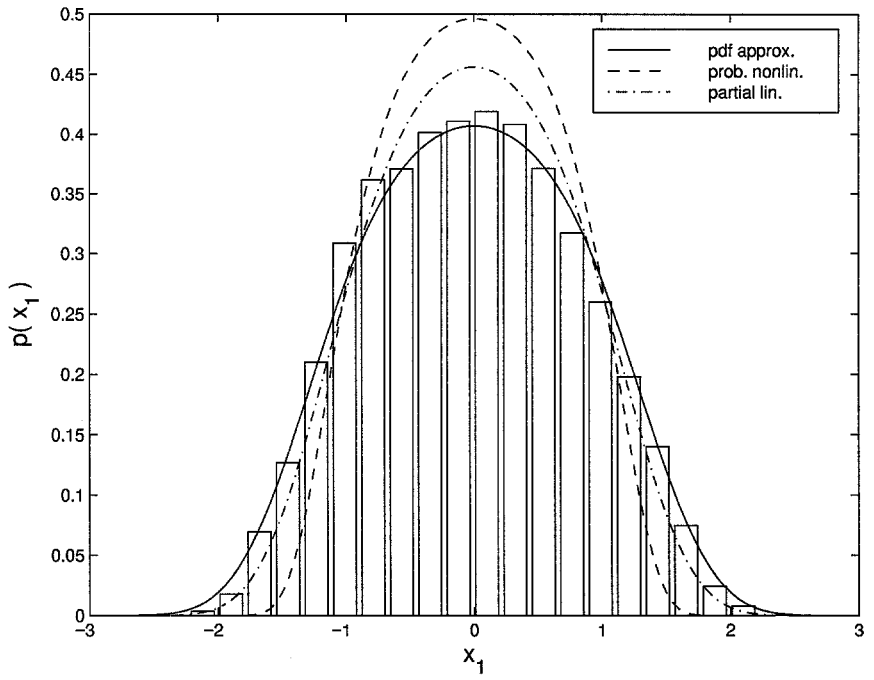


Figure 4.31 Probability density function $p(x_1)$ for the nonlinearly damped Duffing oscillator. The histogram shows results from numerical simulation.

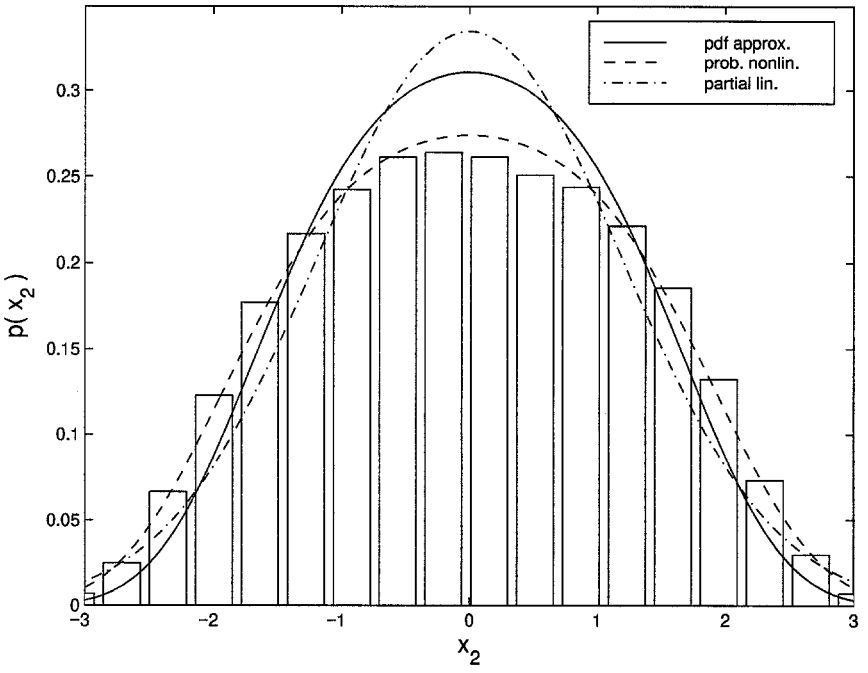


Figure 4.32 Probability density function $p(x_2)$ for the nonlinearly damped Duffing oscillator. The histogram shows results from numerical simulation.

computing outcrossing rates. The outcrossing rates for the safe set

$$\{\mathcal{S} = x \in \mathbb{R}^2 : x_1 \in (-b, b)\}$$

computed by the various methods are shown in Figure 4.33. Also shown are estimates obtained by using the method of probabilistic linearization and minimizing the weighted $\mathcal{L}^2$ norm of the Fokker-Planck equation error, where the weighting function is chosen as in the earlier sections on outcrossing rate estimation to place emphasis near $x_1 = \pm b$. The value of the variance parameter in the weighting function is $\sigma_\rho^2 = \frac{3}{4}$.

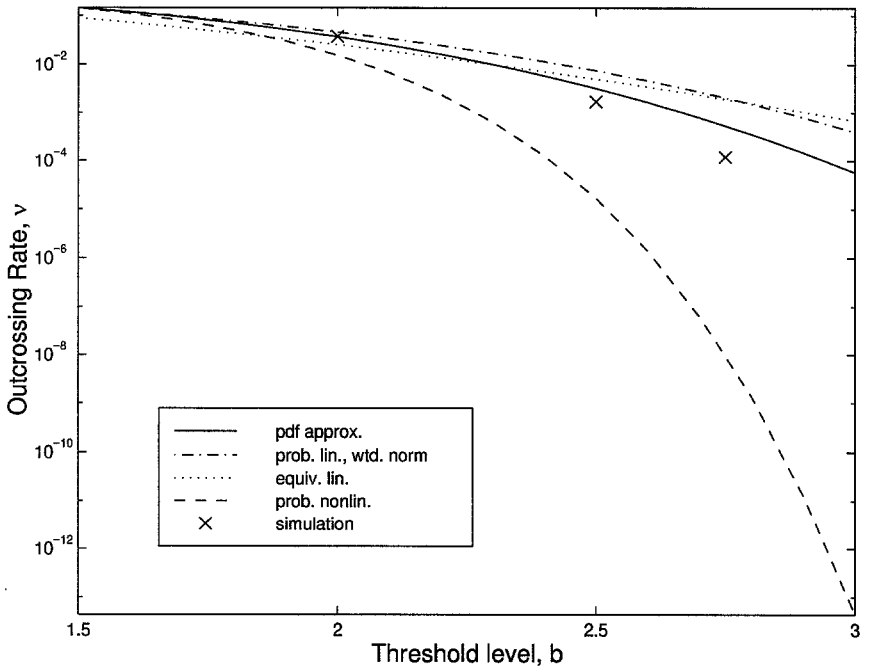


Figure 4.33 Outcrossing rate estimation for the nonlinearly damped Duffing oscillator. $D = \pi, \alpha = 0.5, \beta = 0.1, \gamma = 1, \epsilon = 0.5$

As in example 3, it is observed that the outcrossing rates predicted by probabilistic nonlinearization are far less than those predicted by the other methods. Also, notice that the best agreement with the simulation results is obtained by the

direct approximation of the probability density function. For this example, the outcrossing rate for probabilistic nonlinearization decays as $e^{-c_1 b^8}$ while for the direct probability density function approximation, p_F, the outcrossing rate goes as $e^{-c_2 b^4}$, and as $e^{-c_3 b^2}$ for the method of equivalent linearization.

4.5 Equivalent and Partial Linearization Revisited

Although the methods of equivalent and partial linearization were developed by approximating the stochastic differential equation, it is interesting to see how these methods could be derived from the Fokker-Planck equation and how they compare with the proposed methods.

Consider the single degree-of-freedom oscillator

$$\begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} x_2 \\ -f(x_1, x_2) - g(x_1) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sqrt{2D} \end{pmatrix} dw(t) \tag{4.55}$$

with the associated stationary Fokker-Planck equation

$$L_{nl}p = \left(g(x_1)\frac{\partial p}{\partial x_2} - x_2 \frac{\partial p}{\partial x_1} \right) + \frac{\partial}{\partial x_2}\left(f(x_1, x_2)p + D\frac{\partial p}{\partial x_2} \right) = 0 \tag{4.56}$$

4.5.1 Equivalent Linearization

In the method of equivalent linearization, (4.55) is replaced by a linear system. The corresponding probability density function for the system response is given by

$$p_{lin}(x_1, x_2 | \sigma_{x_1}, \sigma_{x_2}) = \frac{1}{2\pi\sigma_{x_1}\sigma_{x_2}} \exp\left(-\frac{x_1^2}{2\sigma_{x_1}^2} - \frac{x_2^2}{2\sigma_{x_2}^2} \right).$$

Substituting p_{lin} into the Fokker-Planck equation (4.56) gives

$$L_{nl}(x)p_{lin}(x) = \left(\frac{x_1 x_2}{\sigma_{x_1}^2} - \frac{g(x_1)x_2}{\sigma_{x_2}^2} \right) p_{lin} + \frac{\partial}{\partial x_2}\left(\left(f(x_1, x_2) - D\frac{x_2}{\sigma_{x_2}^2} \right) p_{lin} \right).$$

Minimizing the mean-square value of each term in parenthesis, i.e. solving

$$\min_{\sigma_{x_2}} E\left[\left(f(x_1,x_2) - D\frac{x_2}{\sigma_{x_2}^2}\right)^2\right] \quad \text{and then} \quad \min_{\sigma_{x_1}} E\left[\left(\frac{x_1 x_2}{\sigma_x^2} - \frac{g(x)x_2}{\sigma_{x_2}^2}\right)^2\right]$$

gives

$$\begin{aligned}\sigma_{x_2}^2 &= D\,\frac{E[\,x_2^2\,]}{E[x_2\,f(x_1,x_2)\,]}\\ \sigma_{x_1}^2 &= \sigma_{x_2}^2\,\frac{E[\,x_1^2\,]}{E[\,x_1\,g(x_1)\,]}\end{aligned}$$

which are the variances given by the method of equivalent linearization.

4.5.2 Partial Linearization

In the method of partial linearization, the nonlinear system (4.55) is replaced by a linearly damped system

$$\begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} x_2 \\ -\beta_{eq}x_2 - g(x_1) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sqrt{2D} \end{pmatrix} dw(t)$$

which has the corresponding stationary probability density function

$$p_{pl}(x|\beta_{eq}) = c\,\exp\left(-\frac{\beta_{eq}G(x_1)}{D} - \frac{\beta_{eq}x_2^2}{2D}\right)$$

where $G(x_1) = \int_0^{x_1} g(\xi)\,d\xi$ and c is a normalization constant.

Notice that

$$g(x_1)\frac{\partial p_{pl}}{\partial x_2} - x_2\frac{\partial p_{pl}}{\partial x_1} = 0$$

so that error in the stationary Fokker-Planck equation (4.56) reduces to

$$\begin{aligned}L_{nl}p_{pl} &= \frac{\partial}{\partial x_2}\left(f(x_1,x_2)p_{pl} + D\frac{\partial p_{pl}}{\partial x_2}\right)\\ &= \frac{\partial}{\partial x_2}\left\{(f(x_1,x_2) - \beta_{eq}x_2)\,p_{pl}\right\}.\end{aligned}$$

Minimizing the mean square value of the term in parenthesis with respect to β_{eq} gives

$$\beta_{eq} = \frac{E[\, x_2 \, f(x_1, x_2)\,]}{E[\, x_2^2\,]}$$

which is the partial linearization criterion (3.20).

4.6 Summary

Three new methods have been presented for analyzing nonlinear dynamical systems under additive stochastic excitation and several examples were presented illustrating the accuracy of the methods. The first method presented, probabilistic linearization, finds the Gaussian probability density function that minimizes the Fokker-Planck equation error. Since the probability density function for the response of linear systems is Gaussian, this method is expected to work well for systems with small nonlinearities. In addition, it was shown that simple weighting functions can be used to improve the estimates for mean square values and outcrossing rates. The second method presented, probabilistic nonlinearization, finds a non-Gaussian approximation to the stationary probability density function for the response of the system. For the example considered, this method gave a smaller error in the Fokker-Planck equation error than probabilistic linearization and provided better estimates to the mean square values for the response, but provided poor estimates for outcrossing rates. In the third method presented, a different set of approximate non-Gaussian probability density functions than those chosen by the method of probabilistic nonlinearization is used. For the examples considered, this method provided good estimates to both mean square values and outcrossing rates, as well as having the smallest error in the Fokker-Planck equation.

Chapter 5

Modeling Uncertainty

In previous chapters, methods for predicting the response of dynamical systems subjected to uncertain excitation were presented. No uncertainty in the modeling of the system was considered. In this chapter, modeling uncertainties will also be included in the analysis.

The response of dynamical systems is often modeled by a system of ordinary or partial differential equations. The equations are usually obtained from balance laws, experimental observations, or some combination of the two. One source of modeling uncertainty, *prediction-error uncertainty*, is present in any model, as no mathematical model can fully describe the behavior of a physical system. A second source of modeling uncertainty is *parameter uncertainty*. The models typically contain a number of parameters and the values of the parameters that will give the best fit between the response of the model and the physical system are uncertain.

The uncertainty in the parameters will be modeled probabilistically, and typical problems arising when including modeling uncertainty in the analysis involve computing integrals of the form

$$I = \int_{\Theta} f(\theta) p(\theta)\, d\theta \tag{5.1}$$

where $f(\theta)$ and $p(\theta)$ are sufficiently smooth, $\Theta \subset \mathbb{R}^n$, $f(\theta) > 0$ for all $\theta \in \Theta$, and $p(\theta)$ is a probability density function. Two applications in which integrals of the form (5.1) arise are

1. Computing the statistical quantities of interest for an uncertain dynamical system under stochastic excitation. For example, computing the expected outcrossing rate for an uncertain dynamical system is done by choosing $f(\theta) = \nu(\theta)$ where $\nu(\theta)$ represents the conditional outcrossing rate for the system given the parameters θ.

2. Classical reliability integrals. Classical reliability integrals arise in a number of applications; see Madsen et al. (1986) for a number of examples and references. For these integrals, regions of the parameter space (the safe set) are determined to give acceptable performance, while other regions (the unsafe, or failure, set) give unacceptable performance. In this case, it is desired to determine the probability of acceptable [unacceptable] performance. This is done by choosing $f(\theta) \equiv 1$ and Θ to be the safe [failure] set.

In almost all cases where these integrals arise, the integrals cannot be evaluated analytically. Numerical integration can be done to evaluate the integrals, but this is often computationally expensive for a small number of uncertain parameters, and usually prohibitive in more than six or seven dimensions. Another approach is to use Monte Carlo-type methods. Although these methods typically require a large number of samples, the computational cost is virtually independent of the number of uncertain parameters, enabling the methods to be used for large problems. These methods become especially expensive if the function $f(\theta)$ is costly to evaluate or when trying to determine failure probabilities, where the actual value of the integral is often very low, in which case an exceptionally large number of samples is often required.

Another method for approximating integrals of the form (5.1) is to obtain an asymptotic expansion for the integral. This approach has recently been used to determine the outcrossing rates, reliability, and moments for uncertain linear dynamical systems (Papadimitriou et al. 1995, 1997; May 1997) and has shown to give accurate results in a number of examples. The general theory for the asymptotic expansions is given in the next section and the expansion is applied to determine moments and outcrossing rates for nonlinear dynamical systems in section 5.2 and

to evaluate classical reliability integrals in sections 5.4.3 and 5.5.

5.1 Asymptotic Approximation For A Class Of Probability Integrals

Consider the general class of multidimensional integrals of the form (5.1). Papadimitriou, Beck, and Katafygiotis (1995, 1997) have derived an asymptotic approximation based on an expansion of the logarithm of the integrand about the point that corresponds to the maximum of the integrand. The idea is to rewrite the integral (5.1) as

$$I = \int_{\Theta} \exp\left(\ell(\theta)\right)\, d\theta$$

where

$$\ell(\theta) = \ln f(\theta) + \ln p(\theta).$$

Note that since $\exp(\cdot)$ is a monotonic function, the maxima of $\ell(\theta)$ are the same as the maxima of the integrand. Consider first the case for which $\ell(\theta)$ has only one maximum, θ^*, located in the interior of Θ. Expanding $\ell(\theta)$ in a Taylor series about θ^* and using $\nabla\ell(\theta^*) = 0$ gives

$$\ell(\theta) = \ell(\theta^*) - (\theta - \theta^*)^T L(\theta^*)(\theta - \theta^*) + e(\theta)$$

where $L(\theta) = -\nabla\nabla\ell(\theta)$ and $e(\theta) = O\left(\|\theta - \theta^*\|^3\right)$. Substituting the series expansion into the integral (5.1) gives

$$\begin{aligned} I &= \exp(\ell(\theta^*)) \int_{\Theta} \exp\left(-(\theta - \theta^*)^T L(\theta^*)(\theta - \theta^*)\right) \exp(e(\theta))\, d\theta \\ &= f(\theta^*)p(\theta^*) \int_{\Theta} \exp\left(-(\theta - \theta^*)^T L(\theta^*)(\theta - \theta^*)\right) \exp(e(\theta))\, d\theta. \end{aligned}$$

Applying Laplace's method of asymptotic expansion (Bleistein and Handelsman 1986) to the above integral yields

$$I(\theta^*) \sim \frac{(2\pi)^{n/2} f(\theta^*) p(\theta^*)}{\sqrt{\det L(\theta^*)}}. \tag{5.2}$$

Letting λ be the minimum eigenvalue of $L(\theta^*)$, it can be shown that the approximation becomes asymptotically correct as $\lambda \to \infty$. Thus, it is expected that the accuracy of the approximation increases as λ increases, which corresponds to the integrand becoming more "peaked" around the maximum.

If there are a finite number of global maxima of the integrand in Θ, the above procedure can still be applied by summing the contributions from each maximum. Letting the maxima be θ_i^*, $i = 1, \ldots k$, the approximation for the integral is given by

$$I = \sum_{i=1}^{k} I(\theta_i^*).$$

A method which can be used to locate the multiple maxima has been developed by Yang and Beck (1998).

5.2 Moments and Outcrossing Rates for Uncertain Nonlinear Dynamical Systems

To determine the moments and outcrossing rates for nonlinear dynamical systems with uncertain parameters, it is first necessary to be able to determine the conditional moments and outcrossing rates for given parameters. The values can be determined exactly only for certain nonlinear systems, as discussed in chapter 3, but they can be determined approximately by the methods of chapter 4 for other nonlinear systems.

Letting $\theta \in \mathbb{R}^m$ be the uncertain parameters with probability density function $p(\theta)$, $E[x_i^n|\theta]$ be the conditional moments and $\nu(\theta)$ the conditional outcrossing rate,

the moments and outcrossing rate for the uncertain system are

$$E[x_i^n] = \int_{\mathbb{R}^m} E[x_i^n|\theta]\, p(\theta)\, d\theta \tag{5.3a}$$

$$\nu = \int_{\mathbb{R}^m} \nu(\theta)\, p(\theta)\, d\theta. \tag{5.3b}$$

5.2.1 Example 1: Uncertain Duffing Oscillator

In this section, the accuracy of the asymptotic approximation for computing the moments and outcrossing rates for the uncertain, linearly damped Duffing oscillator will be investigated. The Duffing oscillator considered is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x + \epsilon x^3 = \sqrt{2D}n(t) \tag{5.4}$$

where $\zeta, \omega_n, \epsilon$, and D are all uncertain and $n(t)$ is zero-mean, Gaussian white noise with $E[n(t)n(t+\tau)] = \delta(\tau)$. The uncertain variables are taken to be independently, lognormally distributed. The lognormal probability density function with mean parameter m_θ and variance parameter σ_θ is given by

$$p_\theta(\theta) = \frac{1}{\sqrt{2\pi}\,\theta\sigma_\theta} \exp\left(-\frac{(\ln\theta - m_\theta)^2}{2\sigma_\theta^2}\right) \text{ for all } \theta > 0. \tag{5.5}$$

The mean parameters for the probability density functions are chosen as

$$m_{\omega_n} = \ln 1 \qquad m_\zeta = \ln 0.5$$
$$m_D = \ln 0.75 \qquad m_\epsilon = \ln 0.25.$$

The mean values for the uncertain variables vary slightly with the variance parameter for the probability density function, but using the above parameters gives

$$E[\omega_n] \approx 1 \qquad E[\zeta] \approx 0.5$$
$$E[D] \approx 0.75 \qquad E[\epsilon] \approx 0.25.$$

The value of the variance parameter, σ, in the lognormal probability density function is approximately the coefficient of variation for the uncertain variable. A common

value of σ will be taken for this parameter and the dependence of the results on σ will be investigated.

The stationary probability density function for the Duffing oscillator (5.4) given the parameters θ is

$$p(x,\dot{x}|\zeta,\omega_n,\epsilon,D) = a(\zeta,\omega_n,\epsilon,D)\exp\left(-\frac{\zeta\omega_n^3x^2}{D} - \frac{\epsilon\zeta\omega_n x^4}{2D} - \frac{\zeta\omega_n\dot{x}^2}{D}\right) \tag{5.6}$$

where $a(\zeta,\omega_n,\epsilon,D)$ is the normalization constant for the probability density function. The conditional moments and outcrossing rates can easily be computed from the probability density function and the moments and outcrossing rate for the uncertain system are given by the integrals (5.3a,5.3b).

The effect of parameter uncertainty on both mean square values and expected outcrossing rates for the Duffing oscillator is illustrated in Tables 5.1–5.5 and Figure 5.1. In order to investigate the sensitivity of the results to the parameters, results are presented for cases when only one or two of the variables are considered to be uncertain, as well as for cases when all of the variables are uncertain. When only some of the variables are considered uncertain, the variables which are not considered to be uncertain are set to their mean values.

Results obtained by the asymptotic approximation are compared with those obtained by evaluating the integrals numerically. For one or two uncertain parameters, the numerical integration is done directly. For more than two uncertain variables, direct numerical integration becomes too computationally expensive, and the method of importance sampling (Schueller and Stix 1987; Bucher 1988) is used to compute the integrals. Importance sampling is a Monte Carlo-type method in which most of the samples are generated in the region where the integrand is largest. The weighting functions used in the importance sampling computations are the same as those used in Papadimitriou et al. (1995).

The results illustrate several points. First, the asymptotic approximation provides good estimates for the integrals even when there is large uncertainty in the parameters. Additionally, it is observed that the outcrossing rate, and hence the

probability of failure, is much more sensitive to parametric uncertainty than the mean square values are. Even for coefficients of variation of 10% on the parameters, the expected outcrossing rate is more than an order of magnitude greater than the outcrossing rate obtained by using the mean values of the parameters. It is also observed that both the mean square values and the outcrossing rates are most sensitive to uncertainty in the natural frequency and least sensitive to uncertainty in the nonlinear stiffness term, ϵ.

Uncertain variable	ζ	D	ϵ	ω_n
Numerical integration	0.54	0.52	0.53	0.57
Asymptotic approximation	0.53	0.51	0.52	0.56

Table 5.1 Mean square values, $E[x^2]$, for the linearly damped Duffing oscillator with one uncertain variable with variance parameter $\sigma = 0.2$. The variables which are not considered as uncertain are set to their mean values. The mean square value with all of the variables set to their mean values, $E[x^2|\bar{\theta}]$, is 0.53.

Uncertain variables	ϵ, ζ	ζ, D	ϵ, ω_n	ω_n, ζ	ω_n, D
Numerical integration	0.54	0.54	0.57	0.58	0.56
Asymptotic approximation	0.52	0.52	0.55	0.56	0.54

Table 5.2 Mean square values, $E[x^2]$, for the linearly damped Duffing oscillator with two uncertain variables with variance parameter $\sigma = 0.2$. The variables which are not considered as uncertain are set to their mean values. The mean square value with all of the variables set to their mean values, $E[x^2|\bar{\theta}]$, is 0.53.

Variance parameter, σ	0.05	0.10	0.15	0.20
Importance sampling	0.55	0.56	0.57	0.58
Asymptotic approximation	0.55	0.55	0.54	0.54

Table 5.3 Mean square values, $E[x^2]$, for the linearly damped Duffing oscillator when all of the variables are uncertain with variance parameter σ.

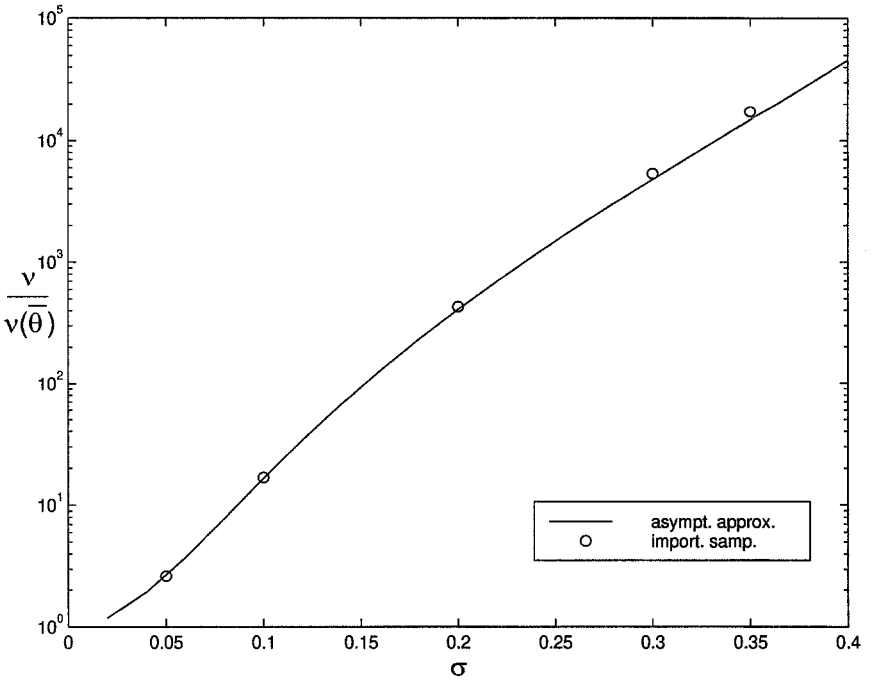


Figure 5.1 Expected outcrossing rates for the Duffing oscillator when all of the variables are uncertain with variance parameter σ. The plot shows the ratio of the expected outcrossing rate ν obtained by evaluating the integral (5.3b) to the outcrossing rate obtained by setting all of the parameters to their mean values, $\nu(\bar{\theta})$.

Uncertain variable	ζ	D	ϵ	ω_n
Numerical integration	7.10×10^{-6}	4.88×10^{-6}	1.34×10^{-6}	6.46×10^{-5}
Asymptotic approximation	7.02×10^{-6}	4.80×10^{-6}	1.27×10^{-6}	6.48×10^{-5}

Table 5.4 Outcrossing rates for the linearly damped Duffing oscillator with one uncertain variable with variance parameter $\sigma = 0.2$. The variables which are not considered as uncertain are set to their mean values. The outcrossing rate with all of the parameters set to their mean values is $\nu(\bar{\theta}) = 6.07 \times 10^{-7}$.

Uncertain variables	ϵ, ζ	ζ, D	ϵ, ω_n	ω_n, ζ	ω_n, D
Num. int.	1.08×10^{-5}	2.14×10^{-5}	9.42×10^{-5}	1.40×10^{-4}	1.12×10^{-4}
Asymptotic	1.06×10^{-5}	2.05×10^{-5}	9.35×10^{-5}	1.40×10^{-4}	1.09×10^{-4}

Table 5.5 Outcrossing rates for the linearly damped Duffing oscillator with two uncertain variables with variance parameter $\sigma = 0.2$. The variables which are not considered as uncertain are set to their mean values. The outcrossing rate with all of the parameters set to their mean values is $\nu(\bar{\theta}) = 6.07 \times 10^{-7}$.

5.2.2 Example 2: Rolling Ship with Uncertain Parameters

The second example is the rolling ship, which was considered earlier is section 4.4.2

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \alpha|\dot{x}|\dot{x} + \omega_n^2 x + \epsilon x^3 = \sqrt{2D}\, n(t).$$

The uncertain parameters for the system are taken to be $\theta = (\zeta, \alpha, \omega_n, \epsilon, D)$.

In this case, the stationary Fokker-Planck equation cannot be solved exactly and the stationary probability density function must be determined approximately to determine the conditional moments and outcrossing rates. It was seen in the previous example that the outcrossing rates are much more sensitive to uncertainty in the parameters than the mean square values are, and for this example, only outcrossing rates will be considered. It was shown in chapter 4 that the outcrossing rates for nonlinear systems can be approximated either by using a weighted norm in the method of probabilistic linearization or by using the direct approximation of the probability density function with a non-Gaussian probability density function, as in section 4.4. For this example, a non-Gaussian probability density function is

chosen as in 4.4.2. The approximate probability density function chosen is

$$p(x,\dot{x}|\phi(\theta),\theta) = a(\theta)\exp\left(-\frac{\phi(\theta)}{D}\left(\frac{\omega_n^2 x^2}{2}+\frac{\epsilon x^4}{4}\right)-\frac{1}{D}\left(\zeta\omega_n\dot{x}^2+\frac{\alpha|\dot{x}|\dot{x}^2}{3}\right)\right)$$

where ϕ is the parameter of the approximate probability density function which was called in θ in section 4.4 and $a(\theta)$ is a normalization constant. Equation (4.48) gives

$$\phi(\theta) = 2\zeta\omega_n + \alpha\frac{E[|\dot{x}|\dot{x}^2]}{E[\dot{x}^2]}.$$

The expected values in the above equation can be computed analytically, but the resulting expressions are rather complicated and are not presented here. The expected outcrossing rate for the threshold $x = b$ and the system parameters θ is given by

$$\nu(\theta) = \int_0^\infty \dot{x}p(b,\dot{x}|\phi(\theta),\theta)\,d\dot{x}. \tag{5.7}$$

The above integral can also be evaluated analytically, again yielding a rather complicated expression. Accounting for the uncertainty, the expected outcrossing rate is

$$\nu = \int_{\mathbb{R}^5} \nu(\theta)p(\theta)\,d\theta$$

where $p(\theta)$ is the probability density function for the uncertain variables. The variables are assumed to be independently, lognormally distributed with parameters

$$\begin{aligned} m_{\omega_n} &= \ln 2\pi \\ m_\zeta &= \ln 0.05 \\ m_\alpha &= \ln 0.5 \\ m_\epsilon &= \ln 0.25 \\ m_D &= \ln 100. \end{aligned}$$

First, the effects of uncertainty in each of the variables individually is investigated. One variable is assumed to be uncertain, while all of the others are set to their mean values. Results obtained with the asymptotic approximation are compared with those obtained by numerical integration in Tables 5.6 and 5.7 for variance parameters of 0.1 and 0.2, respectively. As in the case of the linearly damped Duffing oscillator, it is seen that the asymptotic approximation gives a very good approximation to the integrals. Additionally, it is seen that the outcrossing rate is again most sensitive to uncertainty in the natural frequency, ω_n.

Uncertain variable	ζ	α	D	ϵ	ω_n
Num. int.	2.33×10^{-5}	2.84×10^{-5}	2.85×10^{-5}	2.32×10^{-5}	9.91×10^{-5}
Asymptotic	2.32×10^{-5}	2.83×10^{-5}	2.85×10^{-5}	2.31×10^{-5}	9.90×10^{-5}

Table 5.6 Expected outcrossing rates, ν, for the rolling ship with one uncertain variable with variance parameter $\sigma = 0.1$. The variables which are not considered as uncertain are set to their mean values. The outcrossing rate with all of the parameters set to their mean values is $\nu(\bar{\theta}) = 2.32 \times 10^{-5}$.

Uncertain variable	ζ	α	D	ϵ	ω_n
Num. int.	1.73×10^{-5}	3.64×10^{-5}	3.67×10^{-5}	1.68×10^{-5}	5.91×10^{-4}
Asymptotic	1.70×10^{-5}	3.58×10^{-5}	3.61×10^{-5}	1.65×10^{-5}	5.91×10^{-4}

Table 5.7 Expected outcrossing rates, ν, for one uncertain variable with variance parameter $\sigma = 0.2$. The variables which are not considered as uncertain are set to their mean values. The outcrossing rate with all of the parameters set to their mean values is $\nu(\bar{\theta}) = 1.68 \times 10^{-5}$.

Results for the case when all of the variables are considered uncertain are shown in Figure 5.2. As in the case of the linearly damped Duffing oscillator, it is seen that the asymptotic approximation provides good estimates to the value of the integral, even when there is large uncertainty in all of the parameters.

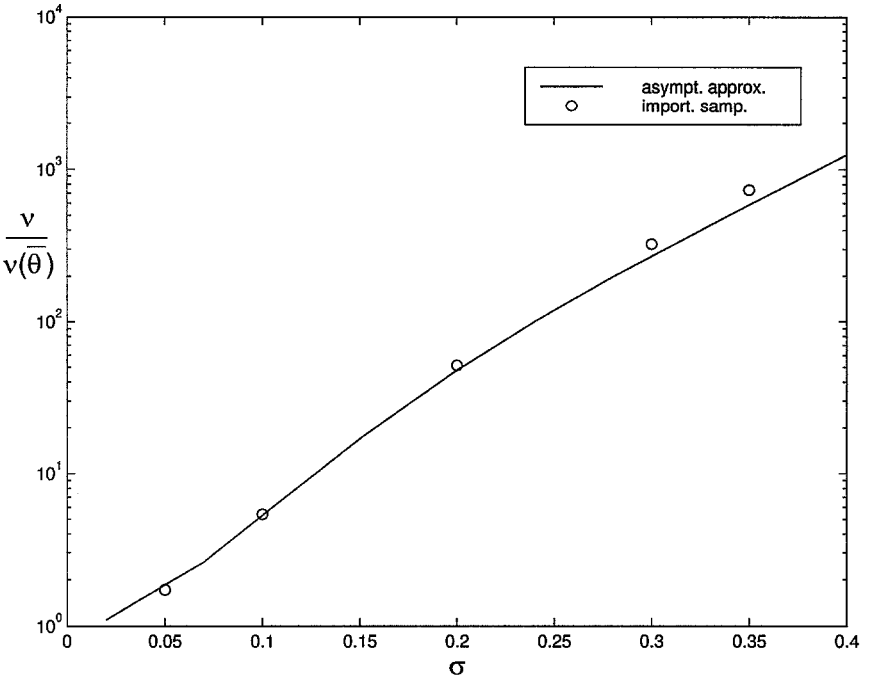


Figure 5.2 Expected outcrossing rates for the rolling ship when all of the variables are uncertain with variance parameter σ. The plot shows the ratio of the expected outcrossing rate ν obtained by evaluating the integral (5.7) to the outcrossing rate obtained by setting all of the parameters to their mean values, $\nu(\bar{\theta})$.

5.3 Classical Reliability Integrals

In classical reliability integrals, the parameter space is divided into two regions, the safe region, $\mathcal{S}$ and the unsafe, or failure, region, $\mathcal{F}$. If the parameter values are in the safe region, the system will have acceptable performance, whereas for parameter values in the unsafe region, the system performance will be unacceptable. The probability density function for the parameters is given by $p(\theta)$ and the goal is to assess the probability of failure, which can be obtained from the integral

$$P_f = \int_{\mathcal{F}} p(\theta)\, d\theta. \tag{5.8}$$

Integrals of the form (5.8) cannot typically be evaluated analytically and numerical evaluation becomes computationally prohibitive as the number of unknown parameters increases. In order to evaluate these integrals, some approximate techniques have been developed. The FORM and SORM methods are reviewed in the next section.

5.4 FORM and SORM Approaches

Two of the most common methods for approximating reliability integrals are the first and second order reliability methods (FORM and SORM, respectively). The first step in either of the methods is to make a transformation of coordinates from the original parameters θ to new variables x so that the new variables are independently, normally distributed, i.e.,

$$p(x) = \phi(x) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}\|x\|^2\right).$$

One such transformation which is applicable to any original distribution $p(\theta)$ is the Rosenblatt transformation (Rosenblatt 1952), as discussed in Madsen et al. (1986).

The second step in the FORM and SORM methods is to approximate the boundary between the safe and unsafe regions by either a first or second order surface. First, consider the case where there is a single surface separating the safe and unsafe

regions in the new variables x. This surface, termed the failure surface, is typically given in the implicit form $g(x) = 0$ and the safe and unsafe sets are the regions

$$\begin{aligned} \mathcal{S} &= \{x \in \mathbb{R}^n : g(x) < 0\} \\ \mathcal{F} &= \{x \in \mathbb{R}^n : g(x) > 0\}. \end{aligned}$$

The probability of failure integral becomes

$$P_f = \int_{g(x)>0} \phi(x)\, dx.$$

This integral can be evaluated analytically only when the failure surface is a hyperplane. In the FORM and SORM methods, the failure surface is approximated by either a first or second order surface in the vicinity of the point on $g(x) = 0$ with minimal distance to the origin. The idea behind these approximations is that $\phi(x)$ decays rapidly as $\|x\|$ increases, so the major contributions to the integral come from the regions in $\mathcal{F}$ closest to the origin. The point on the failure surface closest to the origin in the transformed x-space is called the *design point* and the distance from this point to the origin is often denoted by β.

If there is more than one surface separating the safe and unsafe regions, the surface approximations are made at the design point on each surface, and the integral can be approximated by summing the contributions from each surface, provided that the contribution from the overlapping failure domains is insignificant. A more formal study of system reliability issues can be found in the books by Madsen et al. (1986) and Thoft-Christensen and Murotsu (1986).

5.4.1 FORM

In first-order reliability methods, the failure surface is approximated by a hyperplane tangent to the surface at the point on the surface closest to the origin, as illustrated in Figure 5.3. The approximating hyperplane can be written in the form

$$g_{form}(x) = \langle x, n \rangle - \beta = 0$$

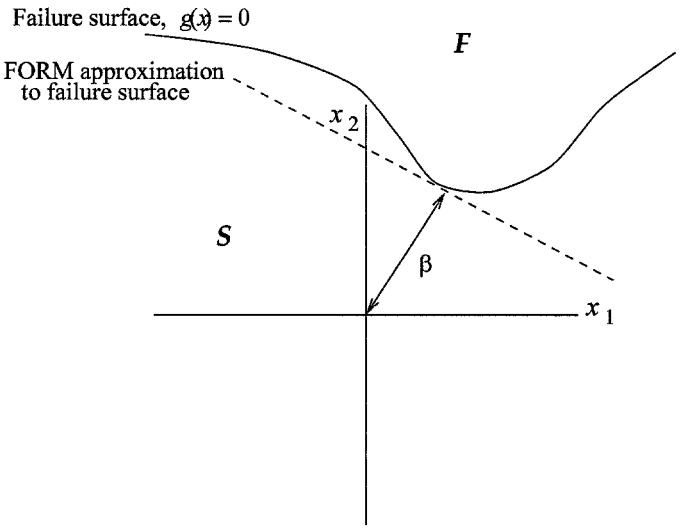


Figure 5.3 FORM approximation to the failure surface

where n is the unit normal to the hyperplane. With this approximation to the failure surface, the integral can be evaluated analytically (Madsen et al. 1986), giving

$$\int_{g_{form}(x)>0} \phi(x)\, dx = \Phi(-\beta) \tag{5.9}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function. This is the value used by the FORM approach to obtain estimates of the failure probability, i.e.,

$$P_f \approx \Phi(-\beta).$$

5.4.2 SORM

In the second-order reliability methods, the curvature of the failure surface at the design point is also accounted for. In this case, a second-order surface is used to approximate the failure surface in the vicinity of the design point, as shown in Figure 5.4.

After a rotation of coordinates, the paraboloid approximation to the failure

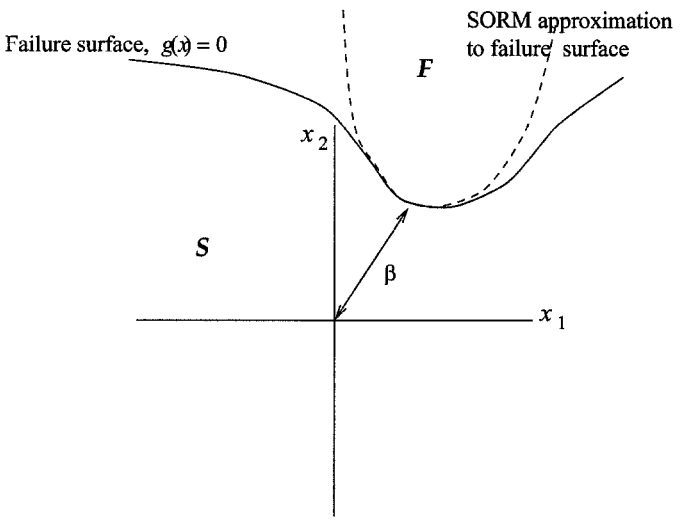


Figure 5.4 SORM approximation to the failure surface

surface can be written in the form

$$g_{sorm}(x) = x_n - \frac{1}{2}\sum_{i=1}^{n-1} a_i x_i^2 - \beta = 0 \tag{5.10}$$

where the a_i's are the principal surface curvatures at the design point and the x_n axis is along the line through the origin and the design point. An alternative procedure for obtaining a second order approximation based on a point-fitting method rather than curvature fitting has been given by Der Kiureghian et al. (1987). In either case, the probability of failure is then approximated by the following integral

$$P_f \approx \int_{g_{sorm}(x)>0} \phi(x)\, dx. \tag{5.11}$$

Unlike the FORM case, the above integral cannot be evaluated analytically, and a further approximation is required. A few different asymptotic approximations have been previously developed for approximating integrals of the form (5.11) (Tvedt 1983; Breitung 1984; Köylüoğlu and Nielsen 1994). In Breitung's approximation,

the asymptotic expansion obtained is

$$P_f(\beta) \sim \frac{\Phi(-\beta)}{\prod_{i=1}^{n-1} \sqrt{1+\beta a_i}}. \tag{5.12}$$

This approximation is valid for $\beta a_i > -1$ for all i and becomes asymptotically exact as $\beta \to \infty$. Tvedt's approximation consists of a sum of three terms, the first of which is Breitung's formula (5.12) and the second and third terms are somewhat more complicated expressions. Köylüoğlu and Nielsen used asymptotic expansions of the cumulative distribution function to obtain a number of different approximations, depending on the signs of the surface curvatures and the order of derivatives which were matched in the expansion. In the next section, the asymptotic expansion discussed in section 5.1 is used to give a simple derivation for an alternative asymptotic expansion to SORM integrals.

5.4.3 A New Asymptotic Expansion for SORM Integrals

The asymptotic method described in section 5.1, provides a very simple means for obtaining an asymptotic expansion for SORM integrals. The expansion differs from Breitung's formula, although the two formulas become asymptotically equivalent as $\beta \to \infty$.

The SORM integral is written as

$$P_f = \int_{\mathcal{F}} \phi(x)\, dx$$

where $\mathcal{F} = \{x \in \mathbb{R}^n : g_{sorm}(x) > 0\}$. The approximation described in section 5.1 is not directly applicable to this integral, since the point maximizing the integrand (the design point) is on the boundary of $\mathcal{F}$. Recall from (5.10) that the failure surface is given by $x_n = \beta + \frac{1}{2}\sum_{i=1}^{n-1} a_i x_i^2$. In order to simplify the notation in the following, let $y = (x_1, \ldots x_{n-1})$, $m = n-1$, and order the curvatures so that

$a_1 \leq a_2 \leq \ldots \leq a_m$. With this notation, the integral can be rewritten as

$$\begin{aligned} P_f &= \int_{\mathbb{R}^m} \left(\int_{\beta + \frac{1}{2}\sum_{i=1}^m a_i y_i^2}^{\infty} \phi(x_n)\, dx_n \right) \phi(y)\, dy \\ &= \int_{\mathbb{R}^m} \Phi\left(-\beta - \frac{1}{2} \sum_{i=1}^m a_i y_i^2 \right) \phi(y)\, dy. \end{aligned} \tag{5.13}$$

The integral (5.13) is over all of $\mathbb{R}^m$, ensuring the maxima of the integrand will be in the interior and that the asymptotic expansion (5.2) can be applied. For most cases of engineering interest, this is especially easy, as given by the following theorem.

Theorem 5.1 *Applying the asymptotic expansion (5.2) to the integral in (5.13), and considering separately the three cases of system parameters which are distinguished by the number of maxima of the integrand, gives the following:*
Case 1: If $a_1 \frac{\phi(-\beta)}{\Phi(-\beta)} > -1$,

$$P_f \sim \frac{\Phi(-\beta)}{\prod_{i=1}^m \sqrt{1 + a_i \frac{\phi(-\beta)}{\Phi(-\beta)}}}. \tag{5.14}$$

Case 2: If $a_1 \frac{\phi(-\beta)}{\Phi(-\beta)} < -1$ and $a_1 \neq a_2$,

$$P_f \sim 2(1 - \Phi(-\gamma_1 \sqrt{h_{11}})) \frac{\Phi\left(-\beta - a_1\gamma_1^2/2\right) \exp\left(-\gamma_1^2/2\right)}{\sqrt{\det L(y^*)}} \tag{5.15}$$

where $h_{11} = \beta a_1 + a_1^2\gamma_1^2/2 + 1$, γ_1 is the unique positive solution of

$$1 + a_1 \frac{\phi(-\beta - a_1\gamma_1^2/2)}{\Phi(-\beta - a_1\gamma_1^2/2)} = 0$$

and

$$\det L(y^*) = \gamma_1^2 h_{11} \prod_{i=2}^m \left(1 - \frac{a_i}{a_1}\right). \tag{5.16}$$

Case 3: If $a_1 \frac{\phi(-\beta)}{\Phi(-\beta)} < -1$ and $a_1 = a_2 = \ldots = a_k$,

$$P_f \sim \frac{2^{(3-k)/2}\gamma^{k-1}\sqrt{\pi}}{\Gamma(k/2)} \frac{\Phi(-\beta - a_1\gamma^2/2)\exp\left(-\gamma^2/2\right)}{\sqrt{\det L(y^*)}} \tag{5.17}$$

where $h = \beta a_1 + a_1\gamma^2/2 + 1$, γ is the unique positive solution of

$$1 + a_1 \frac{\phi(-\beta - a_1\gamma^2/2)}{\Phi(-\beta - a_1\gamma^2/2)} = \frac{k-1}{\gamma^2}$$

and

$$\det L(y^*) = \frac{1}{\gamma^2}\left(2(k-1) + (k-1-\gamma^2)(k-1-\gamma^2 h)\right) \prod_{i=k+1}^{m} \left(1 - \frac{a_i}{a_1}\right). \tag{5.18}$$

Proof:

The proof of case 1 is presented here and the proofs for cases 2 and 3 are given in appendix B. The first step in the asymptotic method is to find the point(s) maximizing the integrand. In this case, there is a unique maximum, as given by the following lemma, which is proved in appendix B.

Lemma 5.1 *If* $a_1 \frac{\phi(-\beta)}{\Phi(-\beta)} > -1$, *then the integrand in (5.13) possesses a unique maximum,* $y^* = 0$.

The next step is to compute the determinant of the Hessian matrix at y^*. Letting $\ell(y) = \ln \Phi(-\beta - \frac{1}{2}\sum_{k=1}^{m} a_k y_k^2) + \ln \phi(y)$ and $L(y) = -\nabla\nabla\ell(y)$, as in section 5.1, the determinant of $L(y^*)$ can be easily evaluated from the following lemma, which is also proved in appendix B.

Lemma 5.2 *If* $a_1 \frac{\phi(-\beta)}{\Phi(-\beta)} > -1$, *then the determinant of the Hessian matrix* $L(y^*)$ *is given by*

$$\det L(y^*) = \prod_{i=1}^{m} \left(1 + a_i \frac{\phi(-\beta)}{\Phi(-\beta)}\right).$$

Thus, from equation (5.2), the asymptotic expansion is given by

$$P_f(y^*) \sim \frac{(2\pi)^{m/2} f(y^*)\phi(y^*)}{\sqrt{\det L(y^*)}} \tag{5.19}$$

where $f(y) = \Phi(-\beta - \frac{1}{2}\sum_{k=1}^{m} a_k y_k^2)$. Using $y^* = 0$, the following results are easily obtained

$$\begin{aligned} f(y^*) &= \Phi(-\beta) \\ \phi(y^*) &= (2\pi)^{-m/2}. \end{aligned}$$

Substituting the above quantities into (5.19) proves case 1 of the theorem. ■

Note that, except for the correction factor of $(1 - \Phi(-\gamma_1 h_{11}))$ discussed in appendix B, equation (5.17) reduces to equation (5.15) in the special case of $k = 1$. Also notice that as $a_2 \to a_1$ in equation (5.15), P_f becomes unbounded and for a_2 close to a_1 the smaller of (5.15) and (5.17) with $k = 2$ should be used as the approximation. Similarly, if a_3 and a_2 are close to a_1, (5.17) can be used with $k = 3$, and so on for more curvatures close to a_1.

Some comments about the new asymptotic expansion

The formula (5.14) was obtained by a different approach based on a McLaurin series expansion of the cumulative distribution function which was valid only for $a_i > 0$ for all i by Köylüoğlu and Nielsen (1994). Köylüoğlu and Nielsen used a different asymptotic expansion for cases when all of the $a_i < 0$ and a third expansion in the case of some $a_i > 0$ and some $a_i < 0$. Formula (5.14) also appeared in a derivation of an importance sampling expression given by Hohenbichler and Rackwitz (1988).

The expansion developed for case 1 is valid as long as $a_i \frac{\phi(-\beta)}{\Phi(-\beta)} > -1$. For large β, this is similar to the condition that $a_i\beta > -1$, since $\frac{\phi(-\beta)}{\Phi(-\beta)} \sim \beta$ as $\beta \to \infty$. This latter condition is usually satisfied in practice, since no point on the failure surface can be inside a hypersphere of radius β if β is the shortest distance from the failure surface to the origin. Thus, for concave failure surfaces, the magnitude of the surface curvatures at the design point must be less than the curvature of the hypersphere of

radius β, giving the condition $a_i\beta > -1$. However, for finite values of β, there exists a range of curvatures for which $a_i \frac{\phi(-\beta)}{\Phi(-\beta)} < -1$ but $a_i\beta > -1$. For these parameter values, the expansions developed for cases 2 and 3 should be used.

Notice that the new formula given by (5.14) is very similar to Breitung's formula (5.12). The only difference is that the terms $(1 + a_i\beta)$ in Breitung's formula become replaced with $1 + a_i \frac{\phi(-\beta)}{\Phi(-\beta)}$. Additionally, since $\frac{\phi(-\beta)}{\Phi(-\beta)} \sim \beta$ as $\beta \to \infty$, it is seen that the two formulas are asymptotically equivalent in the limit as $\beta \to \infty$. However, for finite values of β, values obtained by the two approximations will differ. Numerical comparisons of the two approaches are presented later in Tables 5.8-5.10.

It is of interest to compare the two approximations (5.12) and (5.14) for small values of surface curvatures. In order to do this, consider the asymptotic approximations to the probability of failure as a function of the surface curvatures

$$P_f(a) = \int_{\mathbb{R}^m} \Phi\left(-\beta - \frac{1}{2}\sum_{k=1}^{m} a_k y_k^2\right) \phi(y)\, dy.$$

In the case where all of the surface curvatures are zero, i.e., $a_i = 0$ for all $i = 1, \ldots, m$, both approximations give $P_f(0) = \Phi(-\beta)$, which is the FORM approximation. Note that this is also the exact answer since with all $a_i = 0$, the failure surface is a hyperplane. In order to investigate the approximations for small curvatures, consider the rate of change of P_f with respect to the curvatures, $\left.\frac{\partial P_f}{\partial a_i}\right|_{a=0}$. The exact value can be computed from

$$\begin{aligned}
\left.\frac{\partial P_f}{\partial a_i}\right|_{a=0} &= \left.\frac{\partial}{\partial a_i}\left(\int_{\mathbb{R}^m} \Phi\left(-\beta - \frac{1}{2}\sum_{k=1}^{m} a_k y_k^2\right) \phi(y)\, dy\right)\right|_{a=0} \\
&= \int_{\mathbb{R}^m} -\frac{y_i^2}{2}\phi(-\beta)\phi(y)\, dy \\
&= -\frac{1}{2}\phi(-\beta)\int_{\mathbb{R}^m} y_i^2\phi(y)\, dy \\
&= -\frac{\phi(-\beta)}{2}.
\end{aligned}$$

For Breitung's approximation, the derivatives are computed as

$$\left.\frac{\partial P_f}{\partial a_i}\right|_{a=0} = -\frac{\beta\Phi(-\beta)}{2}$$

whereas for the new approximation,

$$\left.\frac{\partial P_f}{\partial a_i}\right|_{a=0} = -\frac{\phi(-\beta)}{2}.$$

Notice that the new approximation provides the correct derivative, and thus for small curvatures, the error in $P_f(a)$ is of order a_i^2, while the error in Breitung's approximation is of the order $|a_i|$.

Numerical Results

The following tables compare results obtained by the different approximations for various values of β and surface curvatures. The formulas obtained by the new approach, Breitung's approximation, and Köylüoğlu and Nielsen's one term approximation are all very simple. The results obtained by the new approximation are often more accurate than those obtained from these approximations. Tvedt's approximation and Köylüoğlu and Nielsen's three term approximation tend to give more accurate results than the new approximation, but the formulas for these approximations are also considerably more difficult. The exact values presented in Tables 5.8, 5.9 and 5.11 were obtained by Köylüoğlu and Nielsen (1994) using a Gaussian quadrature numerical integration scheme and the values in Table 5.10 were obtained using numerical integration in Mathematica.

All of the tables are for the case when $n = 3$. Table 5.8 is for the case when both surface curvatures of the failure surface are positive. In this case, Köylüoğlu and Nielsen's one term approximation is the same as the formula for the new approximation. In Table 5.9, both surface curvatures are negative, but small enough so that $a_i\frac{\phi(-\beta)}{\Phi(-\beta)} > -1$. In Table 5.10, the case is considered when one of the negative curvatures is large enough so that $a_1\frac{\phi(-\beta)}{\Phi(-\beta)} < -1$ but small enough so that $a_1\beta > -1$. In this case, the formulas (B.8) and (B.10) are used for the new approximation. For

this case, the results from the new approximation are seen to be considerably better than Breitung's approach, and of comparable or better accuracy to Köylüoğlu and Nielsen's approximation.

In Table 5.11, the results of the approximations are investigated when there is a large negative surface curvature. For the cases considered, $a_1\phi(-\beta)/\Phi(-\beta) < -1$ and $a_1\beta < -1$. While these cases are unlikely to occur in practice, it is interesting to note that the new approximations provide much better (several orders of magnitude for large β) results than the other approximations in this case.

5.5 Application of Asymptotic Approximation in Original Variables

As mentioned in section 5.4, the first step in either the FORM or SORM methods is to make a transformation of variables from the original variables, θ, to new variables x, so that the new variables are independently normally distributed. While this can always be done in principle through the Rosenblatt transformation, in many cases the transformation cannot be performed analytically and must be done numerically, which greatly increases the computational requirements. Additionally, by making the transformation of variables, the sensitivity of results to changes in the original parameters is not clear. For these reasons, it is desirable to be able to obtain approximations in the original variables. Breitung (1989) has developed one such formula, but the resulting expression is rather complicated. In this section, an approximation is obtained by the same method used earlier for the transformed variables, yielding a formula much simpler than Breitung's formula.

In the original variables, the probability of failure is given by

$$P_f = \int_{\mathcal{F}} p_\theta(\theta)\, d\theta \tag{5.20}$$

where $p(\theta)$ is the probability density function for θ. Letting $m = n - 1$, $\psi =$

β	Exact	New Approx.	Breitung	Tvedt	K and N, 1 term	K and N, 3 term
5	4.52×10^{-7}	4.63×10^{-7}	4.78×10^{-7}	4.51×10^{-7}	4.63×10^{-7}	4.53×10^{-7}
4	5.87×10^{-5}	6.06×10^{-5}	6.33×10^{-5}	5.82×10^{-5}	6.06×10^{-5}	5.88×10^{-5}
3	3.01×10^{-4}	3.15×10^{-4}	3.38×10^{-4}	3.00×10^{-4}	3.15×10^{-4}	3.02×10^{-4}
2	6.31×10^{-3}	6.74×10^{-3}	7.58×10^{-3}	6.03×10^{-3}	6.74×10^{-3}	6.35×10^{-3}
1	5.67×10^{-2}	6.28×10^{-2}	7.93×10^{-2}	4.88×10^{-2}	6.28×10^{-2}	5.74×10^{-2}

Table 5.8 Comparison of new approximation with Breitung's asymptotic formula, Tvedt's three term expansion, and Köylüoğlu and Nielsen's one and three term approximations. $a_1 = a_2 = 1$.

β	Exact	New Approx.	Breitung	Tvedt	K and N, 1 term	K and N, 3 term
5	5.73×10^{-7}	5.95×10^{-7}	5.73×10^{-7}	5.75×10^{-7}	4.35×10^{-7}	5.45×10^{-7}
4	5.34×10^{-5}	5.49×10^{-5}	5.28×10^{-5}	5.36×10^{-5}	4.06×10^{-5}	5.23×10^{-5}
3	1.97×10^{-3}	2.01×10^{-3}	1.93×10^{-3}	1.98×10^{-3}	1.79×10^{-3}	1.95×10^{-3}
2	2.94×10^{-2}	2.98×10^{-2}	2.84×10^{-2}	2.95×10^{-2}	2.81×10^{-2}	2.93×10^{-2}
1	1.85×10^{-1}	1.87×10^{-1}	1.76×10^{-1}	1.86×10^{-1}	1.81×10^{-1}	1.85×10^{-1}

Table 5.9 Comparison of new approximation with Breitung's asymptotic formula, Tvedt's three term expansion, and Köylüoğlu and Nielsen's one and three term approximations. $a_1 = a_2 = -0.1$

a_2	Exact	New Approx.	Breitung	Tvedt	K and N, 1 term	K and N, 3 term
-0.48	0.086	0.091	0.569	0.571	0.048	0.052
-0.10	0.055	0.075	0.127	n/a	0.038	0.039
0.50	0.035	0.047	0.080	n/a	0.023	0.036

Table 5.10 Comparison of SORM approximations when $\beta = 2$, $a_1 = -0.48$. Tvedt's approximation gives complex numbers for the cases where "n/a" is given.

β	Exact	New Approx.	K and N, 1 term	K and N, 3 term
5	0.0020	0.0019	5.07×10^{-7}	6.64×10^{-6}
4	0.0061	0.0059	5.18×10^{-5}	0.0004
3	0.0200	0.0187	0.0020	0.0067
2	0.0693	0.0640	0.0063	0.0495
1	0.2185	0.2389	0.1789	0.2012

Table 5.11 Comparison of new approximation with Köylüoğlu and Nielsen's one and three term approximations. $a_1 = -1, a_2 = 1$

$(\theta_1, \ldots, \theta_m)$ and $\alpha = \theta_n$, the probability density function for θ can be rewritten as

$$p_\theta(\theta) = p_{\psi,\alpha}(\psi, \alpha) = p_\alpha(\alpha|\psi)p_\psi(\psi)$$

and the failure surface is $g(\psi, \alpha) = 0$. In terms of α and ψ, the above integral can be rewritten as

$$\begin{aligned} P_f &= \int_{\mathbb{R}^m} \left[\int_{g(\psi,\alpha)>0} p(\alpha|\psi) d\alpha \right] p_\psi(\psi) d\psi \\ &= \int_{\mathbb{R}^m} f(\psi) p_\psi(\psi) d\psi \end{aligned} \tag{5.21}$$

where

$$f(\psi) = \int_{g(\psi,\alpha)>0} p_\alpha(\alpha|\psi)\, d\alpha. \tag{5.22}$$

Applying the asymptotic approximation (5.2) to the integral (5.21) gives

$$P_f \sim \frac{(2\pi)^{m/2} f(\psi^*) p_\psi(\psi^*)}{\sqrt{\det L(\psi^*)}} \tag{5.23}$$

where ψ^* is the point maximizing the integrand in (5.21), $\ell(\psi) = \ln f(\psi) + \ln p_\psi(\psi)$ and $L(\psi) = -\nabla\nabla\ell(\psi)$. If there is more than one maxima of the integrand, the probability of failure can be obtained by summing the contributions from each maximum.

A simpler formula than (5.23) can be obtained if the following two conditions

are satisfied

1. One of the parameters, say θ_n, is independently distributed from the others. In this situation, the probability density function for the parameters becomes $p_\theta(\theta) = p_\psi(\psi)p_\alpha(\alpha)$, where $\alpha = \theta_n$.

2. The failure surface $g(\theta) = g(\psi, \alpha) = 0$ can be written as $\alpha = h(\psi)$, and the safe region is the region $\alpha > h(\psi)$. Note that the implicit function theorem guarantees that this can always be done locally provided that $\frac{\partial g}{\partial \alpha} \neq 0$.

Under these conditions, the probability of failure becomes

$$\begin{aligned} P_f &= \int_{g(\psi,\alpha)>0} p_\psi(\psi)\, p_\alpha(\alpha)\, d\alpha\, d\psi \\ &= \int_{\mathbb{R}^m} \left(\int_{-\infty}^{h(\psi)} p_\alpha(\alpha)\, d\alpha \right) p_\psi(\psi)\, d\psi \\ &= \int_{\mathbb{R}^m} P_\alpha(h(\psi))\, p_\psi(\psi)\, d\psi \end{aligned} \tag{5.24}$$

where $P_\alpha(\cdot)$ is the cumulative distribution function for α. Laplace's method of asymptotic expansion can be applied to the integral (5.24). Letting ψ^* be the value of ψ which maximizes the integrand (assuming there is only one maximum), the asymptotic approximation is given by

$$P_f \sim \frac{(2\pi)^{m/2}\, P_\alpha(h(\psi^*))\, p_\psi(\psi^*)}{\sqrt{\det L(\psi^*)}} \tag{5.25}$$

where $L(\psi) = -\nabla\nabla\ell(\psi)$ and $\ell(\psi) = \ln P_\alpha(h(\psi)) + \ln p_\psi(\psi)$. As usual, multiple maxima can be handled by summing the contributions from each maximum.

5.6 Example: Uncertain Single Degree-Of-Freedom Oscillator

To illustrate the approximation in the original variables, consider the following single degree-of-freedom linear oscillator under stochastic excitation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \sqrt{2D}n(t) \tag{5.26}$$

where $n(t)$ is zero-mean, stationary Gaussian white noise with $E[n(t)n(t+\tau)] = \delta(\tau)$. The parameters ζ, ω_n, and D are uncertain, and probability density functions will be specified for the parameters.

The system performance will be judged based on two different criteria. In the first case, the system performance is taken to be acceptable if the mean square displacement of the response stays below some specified limit, σ^2_{max}. The mean square displacement for the oscillator (5.26) is

$$\sigma_x^2(D, \omega_n, \zeta) = \frac{D}{2\zeta\omega_n^3}$$

and the unsafe region in the parameter space is given by

$$\mathcal{F} = \left\{ (D, \omega_n, \zeta) \in \mathbb{R}^3 : g(D, \omega_n, \zeta) = \frac{D}{2\zeta\omega_n^3} - \sigma^2_{max} > 0 \right\}.$$

In the second case, the system performance is taken to be acceptable if the stationary outcrossing rate for the oscillator is below a specified limit, ν_{max}. For the oscillator (5.26), the outcrossing rate past the threshold $x = b$ is given by

$$\nu(D, \omega_n, \zeta) = \frac{\omega_n}{2\pi} \exp\left(-\frac{b^2\zeta\omega_n^3}{D} \right)$$

and the unsafe region is

$$\mathcal{F} = \left\{ (D, \omega_n, \zeta) \in \mathbb{R}^3 : g(D, \omega_n, \zeta) = \frac{\omega_n}{2\pi} \exp\left(-\frac{b^2\zeta\omega_n^3}{D} \right) - \nu_{max} > 0 \right\}.$$

5.6.1 Probability of exceeding mean square limit

First, consider the case where there only two uncertain parameters, ω_n and ζ, so that the methods can be easily visualized. These parameters are assumed to be independently distributed and the probability density functions are chosen to be lognormal distributions. The mean and variance parameters for the probability density functions are chosen to be $m_\zeta = \ln 0.05$, $m_{\omega_n} = \ln 2\pi$, $\sigma_\zeta = 0.15$, and $\sigma_{\omega_n} = 0.05$. The probability density functions are shown in Figure 5.5. The parameter D is assumed to be known, with $D = 2$.

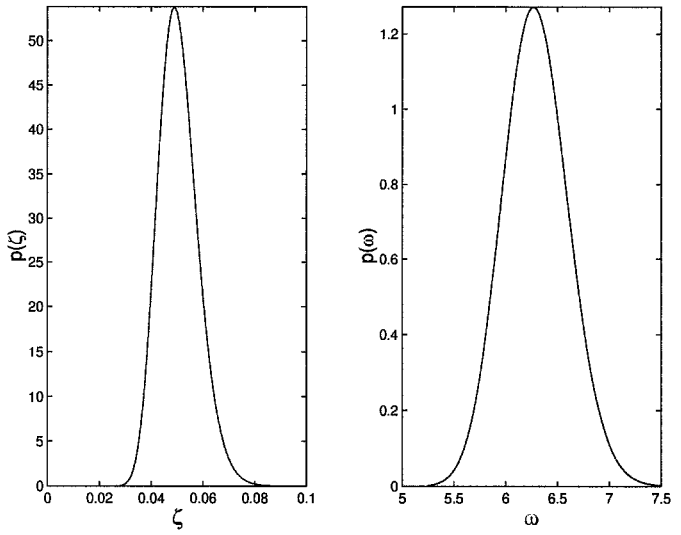


Figure 5.5 Lognormal probability density functions for ζ and ω_n.

For computing the probability that the mean square displacement is below a specified limit, σ^2_{max}, the unsafe region is

$$\mathcal{F} = \left\{ (\omega_n, \zeta) \in \mathbb{R}^2 : g(\omega_n, \zeta) = \frac{D}{2\sigma^2_{max}} - \zeta\omega_n^3 > 0 \right\}$$

which is shown in Figure 5.6 for the case of $\sigma^2_{max} = 0.15$. The probability of failure is given by (5.20) with $\theta = (\zeta, \omega_n)$ and $p(\theta) = p_\zeta(\zeta)p_{\omega_n}(\omega_n)$. Letting $\alpha = \zeta$, $\psi = \omega_n$ and $h(\psi) = D/(2\sigma^2_{max}\psi^3)$, the unsafe region is given by $\alpha < h(\psi)$ and

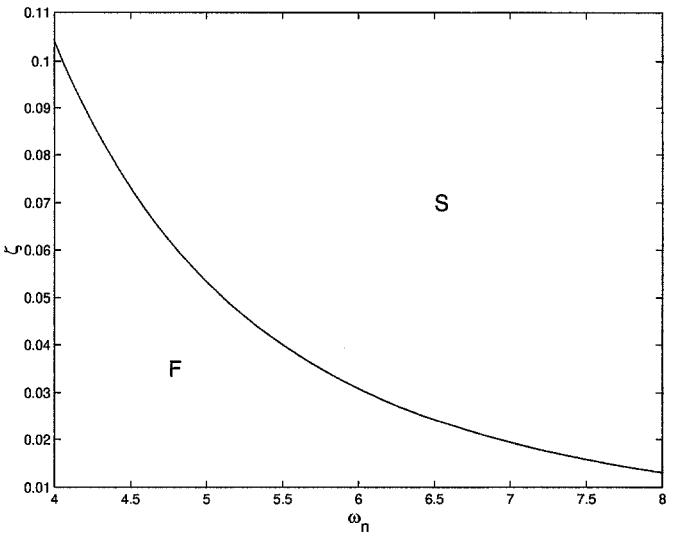


Figure 5.6 Safe and unsafe regions for $\sigma_x^2 < 0.15$

the failure probability can be obtained from (5.25), where $P_\alpha(\cdot)$ is the cumulative distribution function for ζ. The approximation can be easily obtained by solving a one-dimensional unconstrained minimization problem to find the value of ω_n maximizing the integrand.

Note that a different approximation can be obtained by applying (5.25) with $\alpha = \omega_n$, $\psi = \zeta$ and $h(\psi) = \left(D/(2\psi\sigma_{max}^2)\right)^{1/3}$. In the examples considered, it was found that similar results were obtained by using either $\alpha = \zeta$ or $\alpha = \omega_n$ and results are presented only for the asymptotic expansion applied to the integral when $\alpha = \zeta$.

Results are compared with those obtained by Breitung's method and the exact values in Table 5.12 for various values of σ_{max}^2. Breitung's method requires solving a two-dimensional constrained minimization problem and then performing a number of algebraic computations. Since the variables (ω_n, ζ) are independently, lognormally distributed, the transformation of variables to independent normal variables (x_1, x_2)

	Probability of failure		
σ^2_{max}	New method	Breitung's formula	Exact
0.10	1.54×10^{-1}	2.72×10^{-1}	1.55×10^{-1}
0.15	1.71×10^{-3}	1.97×10^{-3}	1.71×10^{-3}
0.20	9.23×10^{-6}	1.00×10^{-5}	9.24×10^{-6}
0.25	4.79×10^{-8}	5.07×10^{-8}	4.79×10^{-8}

Table 5.12 Probability of mean square displacement exceeding σ^2_{max} when ω_n and ζ are lognormally distributed.

can be easily performed. The transformation is given by

$$\begin{aligned} x_1 &= \frac{1}{\sigma_{\omega_n}} \ln \frac{\omega_n}{2\pi} \\ x_2 &= \frac{1}{\sigma_\zeta} \ln \frac{\zeta}{.05} \end{aligned}$$

where σ_{ω_n} and σ_ζ are the variance parameters of the lognormal distributions $p_{\omega_n}(\omega_n)$ and $p_\zeta(\zeta)$. After a little algebra, the unsafe region can be written in terms of the new variables as

$$\mathcal{F} = \left\{ x \in \mathbb{R}^2 : 3\sigma_{\omega_n} x_1 + \sigma_\zeta x_2 < \ln \frac{5\sigma^2_{max}}{2\pi^3} \right\}.$$

Notice that in the transformed variables, the failure surface is a line; therefore, the exact answer can be determined to be

$$P_f = \Phi\left(\frac{1}{\sqrt{\sigma^2_\zeta + 9\sigma^2_{\omega_n}}} \ln \frac{5}{2\pi^3 \sigma^2_{max}} \right).$$

A second case is also considered for which ζ is lognormally distributed as before, but the probability density function for ω_n is as shown in Figure 5.7. In this case, the transformation of variables is not as easy as when ω_n was lognormally distributed, and the FORM approach will not yield the exact answer. The failure surface in the transformed variables is computed numerically and is shown in Figure 5.8 for the case of $\sigma^2_{max} = 0.20$. Results obtained by the various methods are shown in Ta-

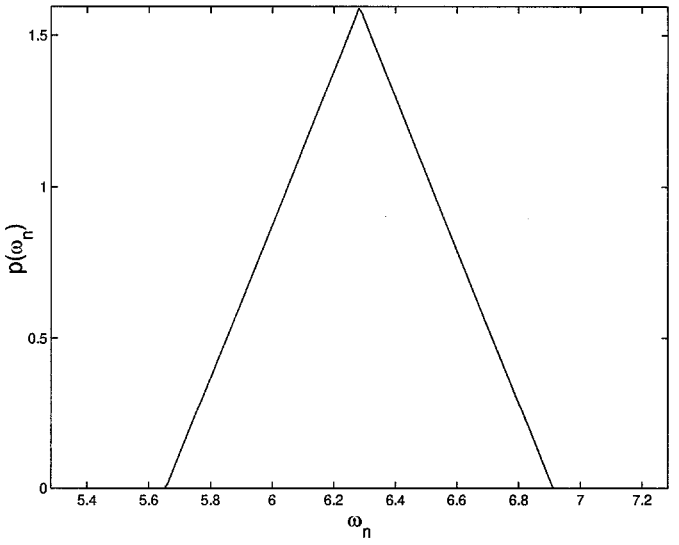


Figure 5.7 Probability density function $p_{\omega_n}(\omega_n)$

ble 5.13, along with the exact answers obtained by numerical integration. For both of the probability density functions investigated, it is seen that the new approximation gives more accurate results than those obtained by Breitung's approximation, especially for relatively large failure probabilities. The FORM approach gives the exact result when ω_n is lognormally distributed, but less accurate results than the proposed approximation when ω_n is distributed as in Figure 5.7.

Next, uncertainty in the excitation amplitude, D, is also considered. The parameter D is taken to be lognormally distributed, with $m_D = \ln 2$ and $\sigma_D = 0.1$ and the probability density function is illustrated in Figure 5.9. Results will be presented only for the case when ω_n and ζ are lognormally distributed, as in Figure 5.5. The resulting estimates for the probability of exceeding various bounds on the mean square amplitude are shown in Table 5.14. As in the case when only ω_n and ζ were uncertain, the failure surface in the transformed variables is a plane, and the exact answer can be obtained. The new approximation again gives more accurate results than those obtained by Breitung's approximation, especially for relatively

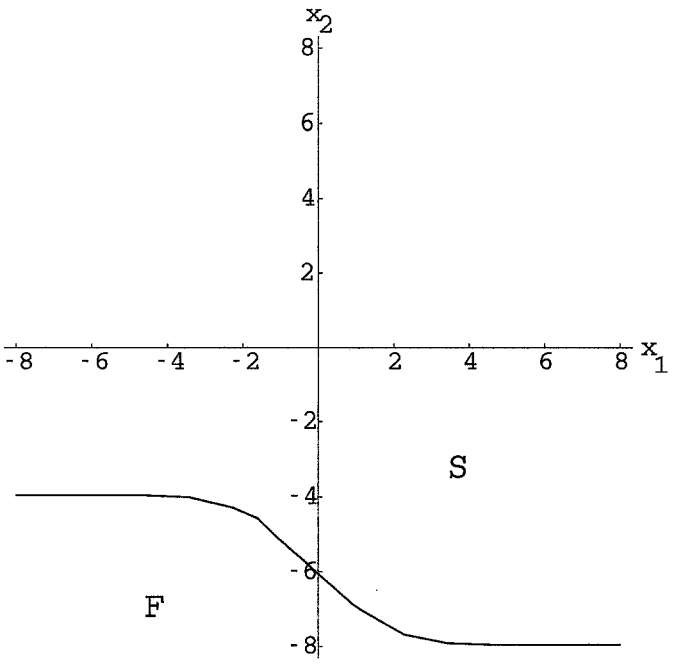


Figure 5.8 Safe and unsafe regions in the transformed variables when ω_n is distributed as in Figure 5.7

	Probability of failure			
σ^2_{max}	New method	Breitung's formula	FORM	Exact
0.10	1.56×10^{-1}	2.92×10^{-1}	1.41×10^{-1}	1.38×10^{-1}
0.15	5.49×10^{-3}	7.50×10^{-3}	7.46×10^{-3}	5.58×10^{-3}
0.20	3.87×10^{-7}	4.56×10^{-7}	6.38×10^{-7}	4.08×10^{-7}
0.25	1.59×10^{-10}	1.70×10^{-10}	2.80×10^{-10}	1.64×10^{-10}

Table 5.13 Probability of mean square displacement exceeding σ^2_{max} when ζ is lognormally distributed and ω_n is distributed as in Figure 5.7.

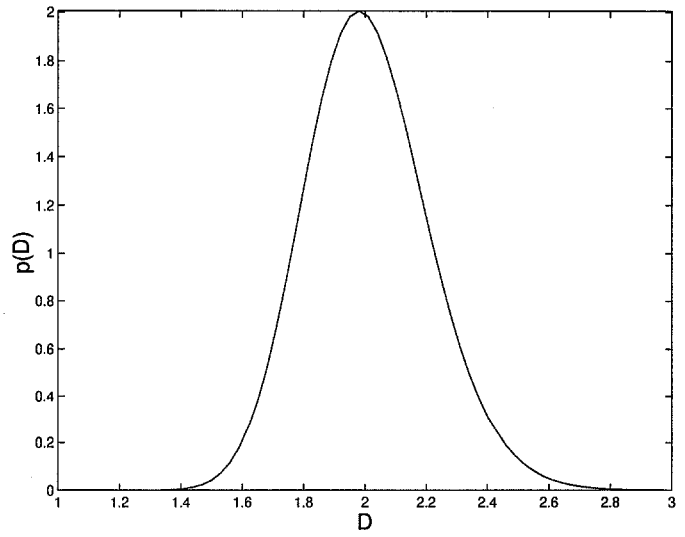


Figure 5.9 Probability density function for D.

large failure probabilities.

5.6.2 Probability of exceeding outcrossing rate limits

The first case considered is when only ω_n and ζ are uncertain and lognormally distributed, as shown in Figure 5.5. For computing the probability that the outcrossing

	Probability of failure		
σ^2_{max}	New method	Breitung's formula	Exact
0.10	1.76×10^{-1}	3.10×10^{-1}	1.80×10^{-1}
0.15	4.02×10^{-3}	4.62×10^{-3}	4.06×10^{-3}
0.20	5.32×10^{-5}	5.73×10^{-5}	5.36×10^{-5}
0.25	6.94×10^{-7}	7.31×10^{-7}	6.99×10^{-7}

Table 5.14 Probability of mean square displacement exceeding σ^2_{max} when D, ω_n, and ζ are lognormally distributed.

rate exceeds a specified limit, ν_{max}, the unsafe region is

$$\mathcal{F} = \left\{ (\omega_n, \zeta) \in \mathbb{R}^2 : g(\omega_n, \zeta) = \frac{\omega_n}{2\pi} \exp\left(-\frac{b^2 \zeta \omega_n^3}{D} \right) - \nu_{max} > 0 \right\}$$

which is illustrated in Figure 5.10 for the case when the threshold $b = 2$ and $\nu_{max} = 10^{-5}$.

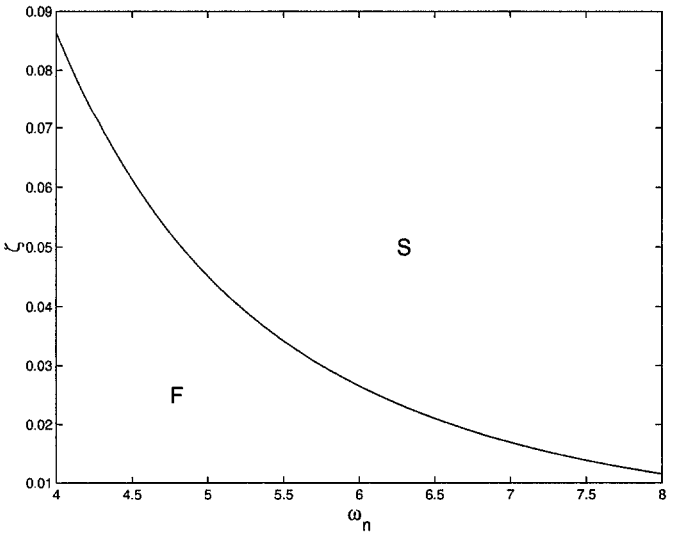


Figure 5.10 Safe and unsafe regions for $\nu_{max} = 10^{-5}$.

The probability of failure can again be approximated by (5.25) with $\alpha = \zeta$, $\psi = \omega_n$, and $h(\psi) = D/(b^2\psi^3)\ln(\omega_n/(2\pi\nu_{max}))$. Results are compared with Breitung's formula and the FORM approach in Tables 5.15 and 5.16. It is again seen that the new approximation gives more accurate results than Breitung's approximation. In applying the FORM approach for this example with both variables lognormally distributed, the failure surface is not a line, but can be very well approximated by a line in the vicinity of the design point, as illustrated in Figure 5.11. For this reason, the FORM approach gives very accurate estimates, as seen in Table 5.15. However, the FORM results are not as accurate when ω_n is distributed as in Figure 5.7, as illustrated by the results in Table 5.16.

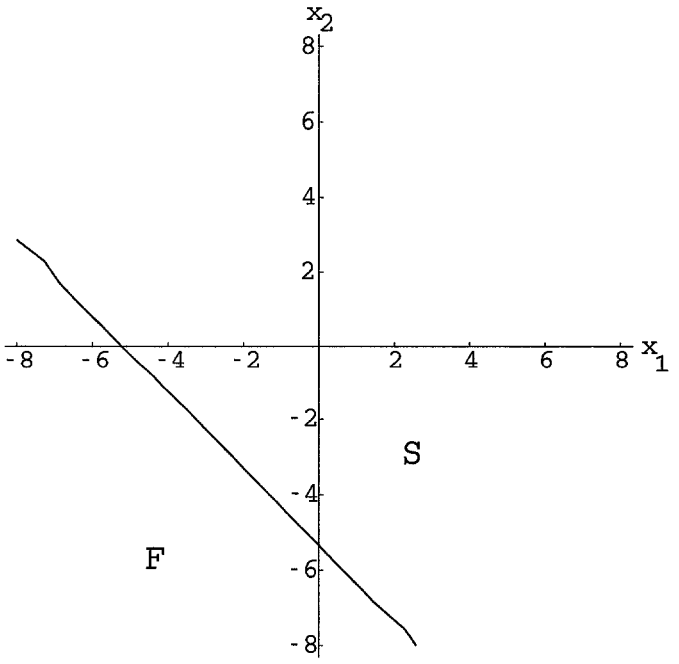


Figure 5.11 Safe and unsafe regions in the transformed variables for $\nu_{max} = 10^{-5}$

	Probability of failure			
ν_{max}	New method	Breitung's formula	FORM	Exact
10^{-8}	7.81×10^{-2}	1.15×10^{-1}	7.85×10^{-2}	7.85×10^{-2}
10^{-7}	2.00×10^{-2}	2.53×10^{-2}	2.00×10^{-2}	2.00×10^{-2}
10^{-6}	2.61×10^{-3}	3.04×10^{-3}	2.62×10^{-3}	2.61×10^{-3}
10^{-5}	1.20×10^{-4}	1.33×10^{-4}	1.21×10^{-4}	1.20×10^{-4}
10^{-4}	9.85×10^{-7}	1.06×10^{-6}	9.86×10^{-7}	9.86×10^{-7}

Table 5.15 Probability of outcrossing rate exceeding specified limit ν_{max} when ζ and ω_n are lognormally distributed.

	Probability of failure			
ν_{max}	New method	Breitung's formula	FORM	Exact
10^{-8}	6.90×10^{-2}	1.25×10^{-1}	7.37×10^{-2}	6.37×10^{-2}
10^{-7}	1.31×10^{-2}	2.11×10^{-2}	1.45×10^{-2}	1.27×10^{-2}
10^{-6}	1.00×10^{-3}	1.39×10^{-3}	1.27×10^{-3}	1.01×10^{-3}
10^{-5}	1.66×10^{-5}	2.07×10^{-4}	2.30×10^{-5}	1.73×10^{-5}
10^{-4}	1.99×10^{-8}	2.28×10^{-8}	3.39×10^{-8}	2.10×10^{-8}

Table 5.16 Probability of outcrossing rate exceeding specified limit ν_{max} when ζ is lognormally distributed and ω_n is distributed as in Figure 5.7.

	Probability of failure			
ν_{max}	New method	Breitung's formula	FORM	Exact
10^{-8}	9.90×10^{-2}	1.46×10^{-1}	1.01×10^{-1}	1.01×10^{-1}
10^{-7}	3.14×10^{-2}	3.99×10^{-2}	3.19×10^{-2}	3.19×10^{-2}
10^{-6}	5.80×10^{-3}	6.75×10^{-3}	5.86×10^{-3}	5.86×10^{-3}
10^{-5}	4.59×10^{-4}	5.06×10^{-4}	4.63×10^{-4}	4.63×10^{-4}
10^{-4}	8.96×10^{-6}	9.53×10^{-6}	9.03×10^{-6}	9.03×10^{-6}

Table 5.17 Probability of outcrossing rate exceeding ν_{max} when D, ω_n, and ζ are lognormally distributed.

Next, uncertainty in D is also considered and the probability function for D is shown in Figure 5.9. Results obtained by the various methods are compared with the exact answer obtained by numerical integration in Table 5.17. Again, the failure surface in the transformed variables is very nearly a line, and the FORM approach gives very accurate estimates. The new asymptotic approximation again gives very good results, even for large failure probabilities.

Chapter 6

Conclusions

Several methods have been presented for analyzing uncertain dynamical systems. Computationally efficient methods have been developed for analyzing nonlinear dynamical systems under stochastic excitation and for evaluating the multi-dimensional integrals which arise when studying systems with modeling uncertainties. All of the methods give approximate solutions and examples have been presented illustrating the accuracy of the approximations.

In chapter 4, three new methods were developed for estimating the stationary probability density function and statistical quantities of interest for the response of nonlinear dynamical systems under uncertain excitation. The methods are all based on obtaining approximate solutions to the Fokker-Planck equation for the system and differ from traditional methods, which are based on finding approximate solutions to the stochastic differential equations for the system.

In the first method presented, probabilistic linearization, the Gaussian probability density function which best solves the Fokker-Planck equation for the nonlinear system of interest is found and taken as an approximation for the probability density function for the nonlinear system. For systems with polynomial-type nonlinearities, the method can be very efficiently applied. Additionally, it was shown that by using simple weighting functions, good approximations could be obtained for both the moments and outcrossing rates of nonlinear systems.

The second method presented, probabilistic nonlinearization, finds a nonlinear

system having a known solution to its Fokker-Planck equation whose probability density function provides the best fit to the Fokker-Planck equation for the nonlinear system of interest. The probability density function for the approximate nonlinear system is non-Gaussian. Since the probability density function for nonlinear systems is known to be non-Gaussian, the hope is that the non-Gaussian approximation will provide a better fit to the tails of the distribution than that obtained by Gaussian approximations. For the example considered, the probabilistic nonlinearization method was able to provide good approximations to mean square values for the response as well as obtaining a better fit to the Fokker-Planck equation than what could be obtained by Gaussian probability density functions. However, the approximation did not do a good job in approximating the tails of the probability density function or the stationary outcrossing rate, as illustrated in Figures 4.31 and 4.33.

In order to improve the estimates of the tails of the probability density function and the outcrossing rates, a third approximation method was developed. In this method, the probability density function was directly approximated, without trying to find an "equivalent system." The approximate probability density functions are non-Gaussian, can be easily and efficiently determined, and yield the exact answer for nonlinear systems that have known solutions to the Fokker-Planck equation. For the examples considered, this method gave accurate results for mean square values and outcrossing rates, as well as providing the best fit to the Fokker-Planck equation.

In chapter 5, approximate methods were developed for analyzing systems with modeling uncertainty. The uncertainty is modeled probabilistically and a simple asymptotic approximation is used to approximate the multi-dimensional integrals which arise. The accuracy of the asymptotic expansion was illustrated with several examples.

In section 5.2, the asymptotic method is applied to determine the second moments and outcrossing rates of uncertain nonlinear systems under uncertain excitation. Two examples are given which illustrate the accuracy of the proposed

approximation. It is also observed that the modeling uncertainties have little effect on the mean square values, but a very significant effect on the outcrossing rates, and hence the probability of failure.

The asymptotic approximation is applied to classical reliability integrals in sections 5.4.3 and 5.5. One approach to approximating classical reliability integrals, the SORM approach, is to first transform the variables to independent standard normal variables, then approximate the failure surface in the transformed variables by a paraboloid, and finally approximate the resulting integral in the transformed variables. In section 5.4.3, the asymptotic approximation is shown to yield a simple and accurate formula for approximating the integral in the transformed variables.

The main disadvantage with the SORM approach is that the variables need to be transformed to standard normal variables. While this can always be done in principle, in many cases the transformation can only be done numerically, in which case a substantial computational effort may be required. Additionally, by transforming the variables, the sensitivity of the results to the original variables is not often clear. In order to avoid these difficulties, the asymptotic approximation was applied in the original variables in section 5.5. The approximation developed is considerably simpler than an existing approximation, and an example is presented for which the accuracy of the new approximation compares favorably to that of the existing approximation.

6.1 Future Work

Another area of future research is to try to develop efficient methods for approximating the stationary probability density function for nonlinear multi-degree-of-freedom systems under additive stochastic excitation. In order to do this, a method for determining a good set of approximate probability density functions for multi-degree-of-freedom systems needs to be developed. In addition, computing the norm of the Fokker-Planck equation error can become computationally expensive for nonlinear multi-degree-of-freedom systems, making it difficult to determine the optimal

parameters for the approximate probability density functions. Since computing the norm of the Fokker-Planck equation error requires evaluating a probability integral similar to those studied in chapter 5, the asymptotic approximations developed could be used to efficiently evaluate the norm and thus enable the optimal parameters to be found. A further area of research is finding approximate probability density functions for systems under parametric excitation.

Future work is also required to investigate the accuracy of the asymptotic approximations for integrals in many dimensions. The approximations showed good accuracy for all of the examples studied in this work, but the largest problem considered involved only a five-dimensional integral. Further testing needs to be done to examine how the accuracy of the approximations changes with the dimension of the integral to be approximated.

Appendix A

Another Choice for the Function $Q(x_1)$

This appendix contains an alternative choice for the function $Q(x_1)$ used in the direct probability density function approximation method discussed in section 4.4. The goal is to select $Q(x_1)$ so that the approximate method presented in section 4.4 is capable of producing the the exact answer for systems for which an exact answer is known.

It was shown that choosing

$$Q(x_1) = G(x_1) = \int_0^{x_1} g(\xi)\, d\xi$$

gives the exact answer if the system is linearly damped. However, this approach will not give the exact answer for other cases for which solutions are known. Here, a different choice of $Q(x_1)$ is made which will give the exact solution to a more general set of nonlinear systems with known solutions of the form

$$\begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ -\tilde{f}(H(x_1, x_2))x_2 - g(x_1) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sqrt{2D} \end{pmatrix} dw(t) \tag{A.1}$$

where

$$H(x, \dot{x}) = \frac{\dot{x}^2}{2} + \int_0^x g(\xi) d\xi.$$

Note that (A.1) includes the linearly damped case by selecting $\tilde{f}(H) =$ constant.

For the general second–order nonlinear oscillator of the form

$$\begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ -f(x_1,x_2) - g(x_1) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sqrt{2D} \end{pmatrix} dw(t) \tag{A.2}$$

the approximate probability density functions can be chosen as

$$p(x_1,x_2|\theta) = a \exp\left(-\frac{\theta}{D}Q(x_1)\right) \exp\left(-\frac{1}{D}F(x_1,x_2)\right) \tag{A.3}$$

where

$$Q(x_1) = \int_0^{x_1} \left.\frac{\partial f(\xi,y)}{\partial y}\right|_{y=0} g(\xi)\, d\xi \tag{A.4}$$

$$F(x_1,x_2) = \int_0^{x_2} f(x_1,\eta)\, d\eta. \tag{A.5}$$

The following lemma proves that if the approximate probability density functions are chosen as above and the system is of the solvable type (A.1), then the method will yield the exact solution for the stationary Fokker–Planck equation.

Lemma A.1 *If the nonlinear system (A.2) is of the form (A.1), i.e.*

$$f(x_1,x_2) = \tilde{f}(H(x_1,x_2))x_2$$

then the probability density function given by (A.3), (A.4) and (A.5) gives the exact solution to the stationary Fokker–Planck equation when $\theta = 1$.

Proof:

It will be shown that $p(x_1,x_2|\theta)$ satisfies the stationary Fokker–Planck equation corresponding to the nonlinear system (A.1). First, recall from section 4.4 that

$$\frac{\partial}{\partial x_2}\left(f(x_1,x_2)p(x_1,x_2|\theta) + Dp(x_1,x_2|\theta)\right) = 0. \tag{A.6}$$

Evaluating the derivatives of $p(x_1, x_2|\theta)$ gives

$$\begin{aligned} \frac{\partial p(x_1, x_2|\theta)}{\partial x_2} &= = -\frac{1}{D} f(x_1, x_2) p(x_1, x_2|\theta) \\ &= -\tilde{f}(H(x_1, x_2))\, x_2\, p(x_1, x_2|\theta) \end{aligned} \tag{A.7}$$

and

$$\begin{aligned} \frac{\partial p(x_1, x_2|\theta)}{\partial x_1} &= -\frac{\theta}{D}\left(\frac{dQ(x_1)}{dx_1} + \frac{\partial F(x_1, x_2)}{\partial x_1}\right) p(x_1, x_2|\theta) \\ &= -\frac{\theta}{D}\left(g(x_1)\left.\frac{\partial f(x_1, y)}{\partial y}\right|_{y=0} + \frac{\partial F(x_1, x_2)}{\partial x_1}\right) p(x_1, x_2|\theta). \end{aligned}$$

Now, since $f(x_1, x_2) = \tilde{f}(H(x_1, x_2))x_2$,

$$\left.\frac{\partial f(x_1, y)}{\partial y}\right|_{y=0} = \tilde{f}(H(x_1, 0))$$

and

$$\begin{aligned} \frac{\partial F(x_1, x_2)}{\partial x_1} &= \frac{\partial}{\partial x_1}\int_0^{x_2} \tilde{f}(H(x_1, \eta))\, \eta\, d\eta \\ &= \int_0^{x_2} \frac{\partial}{\partial x_1}\left(\tilde{f}(H(x_1, \eta))\, \eta\right) d\eta \\ &= \int_0^{x_2} \frac{\partial \tilde{f}(H(x_1, \eta))}{\partial H}\, g(x_1)\, \eta\, d\eta \\ &= g(x_1)\int_0^{x_2} \frac{\partial \tilde{f}(H(x_1, \eta))}{\partial H}\, \eta\, d\eta \\ &= g(x_1)\int_0^{x_2} \frac{\partial\, \tilde{f}(H(x_1, \eta))}{\partial \eta}\, d\eta \\ &= g(x_1)\left(\tilde{f}(H(x_1, x_2)) - \tilde{f}(H(x_1, 0))\right) \end{aligned}$$

so that

$$\frac{\partial p(x_1, x_2|\theta)}{\partial x_1} = -\frac{\theta}{D} g(x_1) \tilde{f}(H(x_1, x_2)) p(x_1, x_2|\theta). \tag{A.8}$$

Combining (A.6), (A.7), and (A.8), and setting $\theta = 1$ gives

$$g(x_1)\frac{\partial p(x_1, x_2|\theta)}{\partial x_2} - x_2\frac{\partial p(x_1, x_2|\theta)}{\partial x_1} + \frac{\partial}{\partial x_2}\left(f(x_1, x_2)p + D\frac{\partial p(x_1, x_2|\theta)}{\partial x_2}\right) = 0$$

i.e., $p(x_1, x_2|\theta = 1)$ satisfies the stationary Fokker–Planck equation. ■

Appendix B

Some Technical Results

B.1 Proof of Lemma 5.1

The integrand is given by

$$h(y) = \Phi\left(-\beta - \frac{1}{2}\sum_{k=1}^{m} a_k y_k^2\right)\phi(y)$$

and a necessary condition for a maximum of $h(y)$ is that

$$\frac{\partial h}{\partial y_i} = -y_i\left(1 + a_i \frac{\phi(-\beta - \frac{1}{2}\sum_{k=1}^{m} a_k y_k^2)}{\Phi(-\beta - \frac{1}{2}\sum_{k=1}^{m} a_k y_k^2)}\right) h(y) = 0 \tag{B.1}$$

for all $i = 1, \ldots, m$. One solution is clearly $y = 0$. Assuming that $a_i \neq a_j$ for all $i \neq j$ and noting that $h(y) > 0$ for all $y \in \mathbb{R}^n$, any other solution of (B.1) must be of the form

$$\hat{y} = (0, \ldots, \pm\tilde{y}_r, \ldots, 0)$$

where $\tilde{y}_r$ is such that

$$1 + a_r \frac{\phi(-\beta - \frac{1}{2}a_r\tilde{y}_r^2)}{\Phi(-\beta - \frac{1}{2}a_r\tilde{y}_r^2)} = 0. \tag{B.2}$$

Note that this clearly requires $a_r < 0$. In the statement of Lemma 5.1, it was given that

$$a_r \frac{\phi(-\beta)}{\Phi(-\beta)} > -1 \text{ for all } r = 1, \ldots, m \tag{B.3}$$

and for $\beta \geq 0$ it can be shown that $\forall z > 0$

$$\frac{\phi(-\beta + z)}{\Phi(-\beta + z)} < \frac{\phi(-\beta)}{\Phi(-\beta)}.$$

Combining the above expression with (B.3) and setting $z = -\frac{1}{2}a_r\tilde{y}_r^2$ gives

$$a_r \frac{\phi(-\beta - \frac{1}{2}a_r\tilde{y}_r^2)}{\Phi(-\beta - \frac{1}{2}a_r\tilde{y}_r^2)} > -1$$

for all $\tilde{y}_r \in \mathbb{R}$. Therefore, there can be no solutions to (B.2), and the only stationary point is $y = 0$. In the proof of Lemma 5.2, it is verified that $y = 0$ is actually a maximum.

In the case where $a_i = a_j$ for some $i \neq j$, additional solutions to (B.1) could exist in the form

$$\hat{y} = (0, \ldots, 0, y_i, y_j, 0, \ldots, 0)$$

where $y_i^2 + y_j^2 = \gamma^2$ and

$$1 + a_i \frac{\phi\left(-\beta - \frac{1}{2}a_i\gamma^2\right)}{\Phi\left(-\beta - \frac{1}{2}a_i\gamma^2\right)} = 0.$$

It was shown above that there can be no solutions to the above equation, since $a_i \frac{\phi(-\beta)}{\Phi(-\beta)} > -1$, and hence $y = 0$ is the only stationary point in this case as well.

B.2 Proof of Lemma 5.2

The function $\ell(y)$ is given by

$$\begin{aligned}\ell(y) &= \ln\Phi\left(-\beta - \frac{1}{2}\sum_{k=1}^{m} a_k y_k^2\right) + \ln\phi(y) \\ &= \ln\Phi\left(-\beta - \frac{1}{2}\sum_{k=1}^{m} a_k y_k^2\right) - \frac{1}{2}\|y\|^2 - \frac{m}{2}\ln 2\pi.\end{aligned}$$

Straightforward computations yield

$$\frac{\partial^2 \ell}{\partial y_i \partial y_j} = -\left(1 + a_i \frac{\phi(-\beta - \frac{1}{2}\sum_{k=1}^{m} a_k y_k^2)}{\Phi(-\beta - \frac{1}{2}\sum_{k=1}^{m} a_k y_k^2)}\right)\delta_{ij} + a_i a_j y_i y_j \frac{\phi(-\beta - \frac{1}{2}\sum_{k=1}^{m} a_k y_k^2)}{\Phi(-\beta - \frac{1}{2}\sum_{k=1}^{m} a_k y_k^2)}\left(\beta + \frac{1}{2}\sum_{k=1}^{m} a_k y_k^2 - \frac{\phi(-\beta - \frac{1}{2}\sum_{k=1}^{m} a_k y_k^2)}{\Phi(-\beta - \frac{1}{2}\sum_{k=1}^{m} a_k y_k^2)}\right)$$

which, when evaluated at $y = 0$, simplifies to

$$\left.\frac{\partial^2 \ell}{\partial y_i \partial y_j}\right|_{y=0} = -\left(1 + a_i \frac{\phi(-\beta)}{\Phi(-\beta)}\right)\delta_{ij}. \tag{B.4}$$

Thus, $\nabla\nabla\ell(y^*)$ is a diagonal matrix and each of the diagonal elements is negative, i.e., $\nabla\nabla\ell(y^*)$ is negative definite. Therefore, $y^* = 0$ is a maximum of the integrand, and the determinant of $L(y^*) = -\nabla\nabla\ell(y^*)$ can be determined by inspection of (B.4) to be

$$\det L(y^*) = \prod_{i=1}^{m}\left(1 + a_i \frac{\phi(-\beta)}{\Phi(-\beta)}\right).$$

B.3 Proof of Case 2 of Theorem 5.1

In this case, $y = 0$ is still a stationary point of the integrand, and there are additional stationary points at $y = (0, 0, \pm\gamma_k, 0, \ldots, 0)$ for all k such that $a_k \frac{\phi(-\beta)}{\Phi(-\beta)} < -1$, where γ_k is the unique positive solution of

$$a_k\, \phi(-\beta - a_k\gamma_k^2/2) = -\Phi(-\beta - a_k\gamma_k^2/2). \tag{B.5}$$

It is easily verified that $y^* = (\pm\gamma_1, 0, \ldots, 0)$ are the only stationary points which are maxima. Evaluating the determinant of the Hessian matrix, $L(y^*) = -\nabla\nabla\ell(y^*)$, gives (5.16) and the asymptotic expansion is given by

$$P_F \sim 2\,\frac{\Phi\left(-\beta - \frac{1}{2}a_1\gamma_1^2\right)\exp\left(-\frac{1}{2}\gamma_1^2\right)}{\sqrt{\det L(y^*)}}, \tag{B.6}$$

where the factor of 2 arises from summing the contributions from each maximum. However, in many cases, the contributions from each maximum overlap as shown in Figure B.1 and summing the contributions from each results in overestimating the probability of failure. For the one dimensional case illustrated in Figure B.1, it can be seen that this problem can be accounted for by only integrating the contribution from the peak near $y_1 = +\gamma_1$ from $y_1 = 0$ to $y_1 = \infty$ and integrating the contribution near the peak $y_1 = \gamma_1$ from $y_1 = -\infty$ to $y_1 = 0$. To see how this is done in the one dimensional case, let

$$\ell(y_1) = \ln\Phi\left(-\beta - \frac{1}{2}a_1 y_1^2\right) + \ln\phi(y_1)$$

and γ_1 be the solution to (B.5) and

$$h_{11} = -\left.\frac{\partial^2\ell(y)}{\partial y_1^2}\right|_{y_1=\gamma_1} = \gamma_1^2\left(\beta a_1 + \frac{1}{2}a_1^2\gamma_1^2 + 1\right). \tag{B.7}$$

Letting $y_+^* = (\gamma_1, 0, \ldots, 0)$ and $y_-^* = (-\gamma_1, 0, \ldots, 0)$, the approximation to the integral can then be written as

$$\begin{aligned} P_f &\approx \exp(\ell(y_+^*))\int_0^\infty e^{-\frac{h_{11}(y_1-\gamma_1)^2}{2}}\,dy_1 + \exp(\ell(y_-^*))\int_{-\infty}^0 e^{-\frac{h_{11}(y_1+\gamma_1)^2}{2}}\,dy_1 \\ &= 2\,\frac{\Phi(-\beta - a_1\gamma_1^2)\exp(-\gamma_1^2)}{\sqrt{h_{11}}}\left(1 - \Phi\left(-\gamma_1\sqrt{h_{11}}\right)\right) \end{aligned}$$

which is just (B.6) multiplied by the additional term $\left(1 - \Phi\left(-\gamma_1\sqrt{h_{11}}\right)\right)$.

Since the Hessian matrix is diagonal, the same approach illustrated for the one dimensional case can easily be extended to the m-dimensional case. The variables $(y_2, \ldots y_m)$ are still integrated from $-\infty$ to ∞, and the y_1 variable is treated as

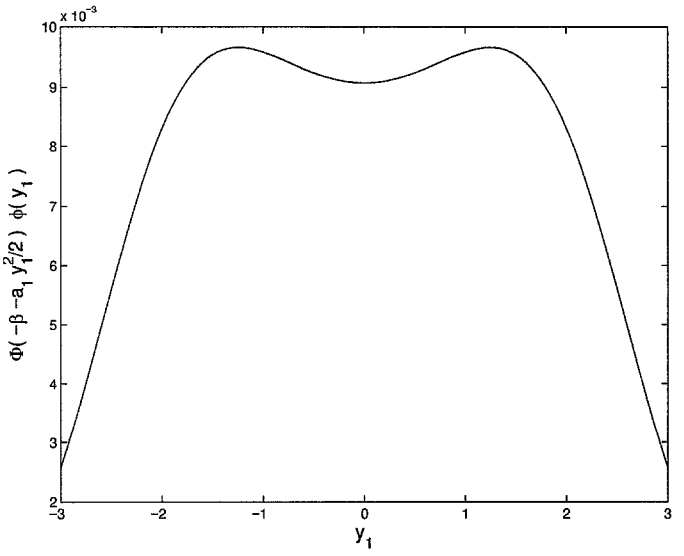


Figure B.1 Integrand for the probability of failure integral when $\beta = 2$ and $a_1 = -0.48$.

above. Evaluating the integral gives

$$P_F \sim 2 \frac{\Phi\left(-\beta - \frac{1}{2}a_1\gamma_1^2\right) \exp\left(-\frac{1}{2}\gamma_1^2\right)}{\sqrt{\det L(y^*)}} \left(1 - \Phi\left(-\gamma_1\sqrt{h_{11}}\right)\right) \tag{B.8}$$

where $\det L(y^*)$ is given by (5.16) and h_{11} by (B.7).

B.4 Proof of Case 3 of Theorem 5.1

In this case, there is a surface of maxima of the integrand given by

$$\mathcal{S}_{max} = \{y = (y_1, y_2, \ldots, y_k, 0, \ldots, 0) : \|y\|^2 = \gamma^2\}$$

where γ is such that

$$1 + a_1 \frac{\phi(-\beta - a_1\gamma^2/2)}{\Phi(-\beta - a_1\gamma^2/2)} = 0.$$

Since the Hessian matrix is singular at all of these points, the previous approach will fail. However, a change of coordinates can be made to avoid this difficulty. Let $r^2 = y_1^2 + \ldots y_k^2$ and notice that the terms in the integrand become

$$\begin{aligned} \tilde{F}(r,y) &= \Phi\left(-\beta - \frac{1}{2}a_1 r^2 - \frac{1}{2}\sum_{i=k+1}^{n} a_i y_i^2\right) \\ \tilde{\phi}(r,y) &= \frac{\phi(r)\phi(y_{k+1})\cdots\phi(y_n)}{(2\pi)^{(k-1)/2}}. \end{aligned}$$

The failure integral becomes

$$P_f = A_{k-1}\int_{\mathbb{R}^{n-k}}\int_0^\infty \tilde{F}(r,y)\tilde{\phi}(r,y)\, r^{k-1}\, dr\, dy_{k+1}\cdots dy_n \tag{B.9}$$

where A_k is the surface area of the unit sphere in k-dimensions, given by (Flanders 1963)

$$A_{k-1} = \frac{2\pi^{k/2}}{\Gamma(k/2)}.$$

The integrand in (B.9) has a unique maximum at $(r^*, y^*) = (\gamma, 0)$, where γ is the unique positive solution of

$$1 + a_1\frac{\phi(-\beta - a_1\gamma^2/2)}{\Phi(-\beta - a_1\gamma^2/2)} = \frac{k-1}{\gamma^2}$$

and the asymptotic method can be applied as before. The Hessian matrix is evaluated to give

$$\det L(y^*) = \frac{1}{\gamma^2}\left(2(k-1) + (k-1-\gamma^2)(k-1-\gamma^2 h)\right)\prod_{i=k+1}^{m}\left(1 - \frac{a_i}{a_1}\right).$$

Combining the results gives

$$P_F \sim \frac{2^{(3-k)/2}\gamma^{k-1}\sqrt{\pi}}{\Gamma(k/2)}\,\frac{\Phi(-\beta - \frac{1}{2}a_1\gamma^2)\,\exp\left(-\frac{1}{2}\gamma^2\right)}{\sqrt{\det L(y^*)}}. \tag{B.10}$$

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